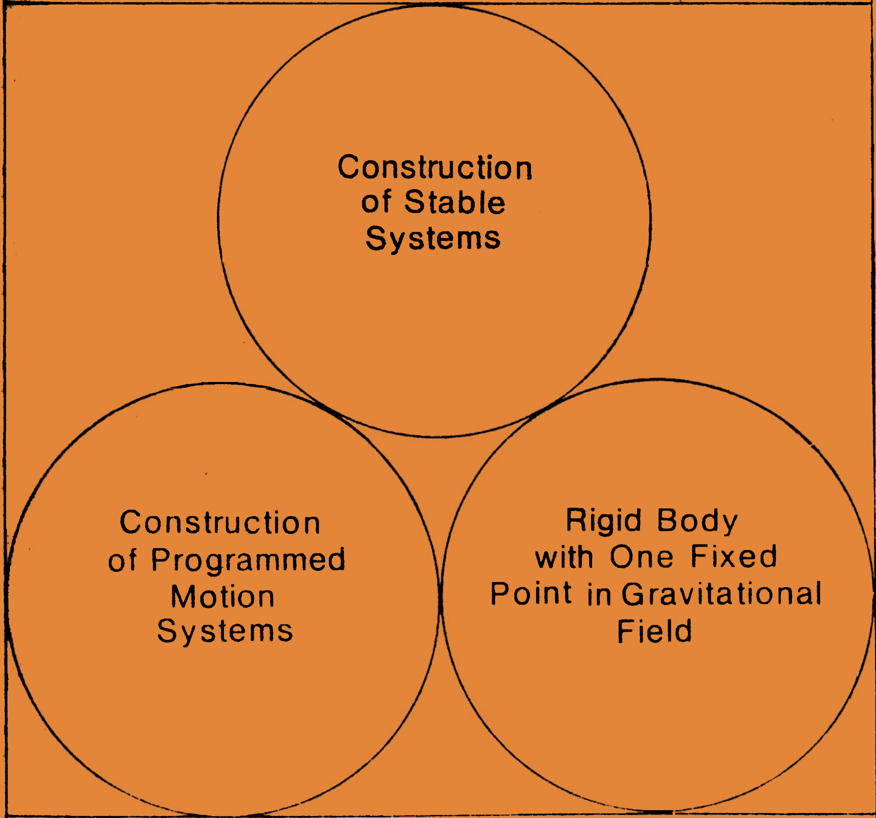
INVERSE PROBLEMS OF DYNAMICS

A. S. Galiullin



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INVERSE PROBLEMS OF DYNAMICS

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Contents

Preface

The definition of the inverse problems of dynamics to which the author of this book adheres is given in G. K. Suslov's work [1]: "By inverse problems of mechanics we mean the determination of forces from the given properties of motion". Classical examples of this kind of problems are the Newton-Bertrand problem on the determination of forces under the action of which a particle moves along conic sections, and the Suslov-Zhukovskii problem on the determination of force functions, admitting the given integrals. The determination of system parameters in such a way that the motion with given properties should be one of the possible motions of the mechanical system in question is also an inverse problem of dynamics. Such problems also include, for example, Meshcherskii's problem on the determination of the change in the mass and the velocity of escaping mass, which is necessary for the motion of a particle with varying mass along a given trajectory in a given force field, and the Chaplygin-Goryachev problem on the determination of the conditions imposed on the mass geometry of a rigid body with one fixed point and on the forces applied to the body in accordance with the given integrals of the appropriate equations of motion.

The analytical construction of a stable mechanical system when its equations of motion are compiled in such a way that the motion of a system having given properties becomes stable is also considered to be an inverse problem.

The analytical construction of mechanical systems of programmed motion where the inverse problems of dynamics are combined with the problem of stability of given properties of motion (programmed motion) turns out to be a generalization of the inverse problems of dynamics.

It should be noted that nowadays many of the inverse problems of dynamics of mechanical systems are already beyond the framework of analytical mechanics. These problems have turned out to be in the nature of signposts and starting points for the modern science of the control of the motion of real systems that have different physical natures and constructions.

This book contains the possible formulations of inverse problems of dynamics, a common technique of solving them based on the construction of equations of motion over a given integral manifold. The analytical construction of stable material systems, as well as programmed systems, is taken into consideration, their motion is described by ordinary differential equations while the properties of motion of the system are assumed to be given in the form of some independent and consistent correlations between the kinematic indices of motion.

The first chapter is devoted to the statement and solution of inverse problems of dynamics and problems of determining generalized forces applied to the mechanical system, its parameters and constraints in accordance with the given properties of its motion. To begin with, a summary of the classical inverse problems of dynamics is given, and the problems are solved. Further on, other versions of the statement of the inverse problems of dynamics are pointed out and the possibility of their solution by reducing them to the problem of constructing the equations of motion of a mechanical system admitting a motion with given properties is considered. Here, the given properties are considered to be an integral manifold of the equations being constructed. The basic inverse problem of dynamics and the problem of restoration and completion of equations of motion are also formulated and solved.

In the second chapter, the analytical construction of stable systems is considered as an inverse problem, namely, the problem of restoration of the equations of motion of a material system, proceeding from the requirement of stability of its given motion after an initial perturbation. First of all, the main ideas and methods of the theory of stability of motion are stated. After this, we consider the application of the method of characteristic numbers and the method of Lyapunov's functions for determining the

parameters of a material system and the forces applied to it in such a way that the given motion is stable and is one of the possible motions of the system in question. The problem concerning the determination of a set of reference functions with respect to which the given motion of the material system is stable is also formulated and solved in this chapter.

In the third chapter, the analytical construction of material systems of programmed motion is considered as an extension of the inverse problems of dynamics with the additional requirement of the stability of the given properties of motion. These problems are the starting points of mathematical problems in programming the motion of material systems with additional forces by varying the parameters of the system during motion, as well as in programming by using control devices. In this chapter a universal scheme is proposed for solving all these problems. This scheme consists in the following: first of all, the appropriate inverse problem is solved. Then, the equations of motion thus obtained are completed by taking into account the requirements of the stability of given properties. From these equations, we determine the unknown controlling forces, parameters of the system and the equations of the control devices. In this chapter the problem of determining the set of functionals stabilized for the programmed motion is formulated and solved.

The fourth chapter is devoted to the inverse problems of dynamics of a rigid body with one fixed point in gravitational field. First, the appropriate Euler dynamic equations are constructed in accordance with tentative particular or first integrals. Then, these equations can be applied to obtain integrable cases by using examples of classical general and particular cases. The application of the equations constructed for solving the problems concerning the control of the motion of a rigid body with one fixed point in gravitational field is also discussed in combination with the problems of stability. In particular, the problem about the realization of a generalized precession of a rigid body with one fixed point in gravitational field is considered.

It can be seen that the book is devoted in general to the formulation and solution of inverse problems of dynamics. It also deals with the analytical construction of stable sys-

tems and the systems of programmed motion with the completeness necessary for interpreting modern problems concerning the control of the motion of material systems in the form of inverse dynamics problems formulated together with the additional requirements.

It should be noted that the construction of differential equations in accordance with given integrals for these equations is the starting point in the formulation of inverse problems of dynamics as well as the problems on controlling the motion of material systems. Way back in 1952, N. P. Erugin [12] published an article devoted to the formulation and solution of this problem. The method set out in his work turned out to be extremely effective in solving various types of inverse problems with a physical content. In the present book, the author has used this method in order to construct the equations of motion of mechanical systems with given kinematic and geometrical properties of motion.

It is also important that a number of specific inverse problems considered in the book have their own histories, and the analysis of the solutions of some of them is a definite landmark in the development of analytical and applied mechanics. The author has refrained from a systematic and exhaustive review of these investigations, assuming that the reader is already familiar with them. These problems are discussed in the book with the only aim of showing the possibility of solving them by a direct application of the technique of constructing the appropriate differential equation in accordance with the given integral manifold.

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Chapter One

INVERSE PROBLEMS OF DYNAMICS

One of the main problems of the dynamics of mechanical systems is to determine the forces and moments from given kinematic elements of motion or, in a more general statement, from given properties of motion. Problems of this kind, along with their various modifications, are called the inverse problems of dynamics [1, 2].

The inverse problems of dynamics have always attracted the attention of mathematicians and engineers primarily because they have a wide scope of application and offer challenging prospects for obtaining a final solution.

The possibility of modelling a number of applied problems, arising in our day-to-day work in the form of inverse problems of dynamics, has led to a considerable broadening of the concept of inverse problems during the past years. Problems have been formulated not only to determine the generalized forces, but also the parameters of a mechanical system, as well as the constraints applied to the system, resulting in the motion of a mechanical system with given properties.

Investigations carried out by mathematicians were stimulated by their inclination to the inverse problems of dynamics; the tendency among engineers towards a mathematical generalization has led to a formulation of the possible inverse problems of dynamics and to quite general methods of solving these problems. Here, if the given properties of motion of a mechanical system can be represented analytically as first integrals or partial integrals of the appropriate equations of motion, then in the general case, the solution of the inverse problems of dynamics is reduced to the construction of differential equations in accordance with the given integrals and the subsequent determination, with

their help, of the unknown forces and moments, parameters and constraints, required for the realization of the motion of a given mechanical system with a priori given properties.

In this chapter, we shall first briefly review the well-known classical inverse problems of dynamics and indicate the methods of solving them. Then, as a generalization of these problems, possible versions of the statement of the inverse problems of dynamics will be formulated. We shall describe a common technique for solving these problems as problems of constructing the equations of motion in accordance with a given integral manifold. Here, the equations of motion will be constructed in the generalized coordinates as well as in canonical variables.

Sec. 1. Classical Inverse Problems

Let us consider some well-known problems of dynamics, which proved to be fundamental in that their appropriate generalization led to the formulation of possible statements of the inverse problems in analytical mechanics.

1.1. Newton's Problem. One of the first inverse problems of dynamics which was solved in the past is Newton's problem on the determination of forces, under the action of which planetary motion takes place with the following properties (Kepler's laws) [3]:

A. The Sun is situated at one of the foci of the ellipses, which are the orbits of planetary motion.

B. The planets have a constant sector velocity.

C. The square of the time taken by a planet to complete a revolution around the Sun is proportional to the cube of the semi-major axis of its orbit.

It is well known that the solution of Newton's problem is obtained in the following way.

From the first two properties of motion, it immediately follows that the force is central, with the centre of field at the focus of the ellipse where the Sun is situated. The magnitude and the direction of the force causing the planet to move with given properties are determined by using the equation of motion of a particle in a central force field (in Binet's form). This force turns out to be attractive, directly proportional to the mass of the planet, and inversely proportional to the square of the distance between the centres

of the Sun and the planet. Further, the third property of the motion of the planet is used to express the constant of proportionality in terms of universal constants (the mass of the Sun and the gravitational constant), and we get the familiar expression for the unknown force.

1.2. Bertrand's Problem. Newton's problem led to subsequent formulations of more general problems concerning the determination of the forces in accordance with the given properties of motion. One such problem is called Bertrand's problem on the determination of the force $\mathbf{F}(X, Y)$ under the action of which a particle moves along a conic section under any initial conditions. The force is supposed to depend only on the position (x, y) of the particle. This problem attracted the attention of a large number of engineers and mathematicians in the last century [2, 4, 5] as its statement predetermined the statements of several more general inverse problems of dynamics and its solution included the solution of Newton's problem itself, with less information about the properties of motion of the planets.

It should be noted that mathematically, the solution of Bertrand's problem results in the construction of the right-hand sides of a system of two second-order equations

$$\begin{aligned} \ddot{x} &= X(x, y), \\ \ddot{y} &= Y(x, y) \end{aligned} \tag{1.2.1}$$

for which a conic section is the integral curve.

Let us write the equation of a conic section relative to the coordinate system where the origin of coordinates coincides with one of the foci, and the x-axis coincides with the symmetry axis of the conic section,

$$\sqrt{x^2 + y^2} = ex + p, \tag{1.2.2}$$

where e is the eccentricity of the conic section, and p is half the latus rectum.

Differentiating Eq. (1.2.2) twice with respect to time, and taking into account Eqs. (1.2.1), we obtain

$$\frac{(x\dot{y} - \dot{x}y)^2}{r^3} + \frac{x}{r} X + \frac{y}{r} Y = eX \quad (r = \sqrt{x^2 + y^2}). \tag{1.2.3}$$

Using the first derivative of (1.2.2) with respect to time, we can obtain the following equality from (1.2.3):

$$\frac{(x\dot{y}-\dot{x}y)^2}{r^2}=\frac{y}{\dot{x}}(\dot{y}X-\dot{x}Y), \tag{1.2.4}$$

which takes place at any moment of time under any initial conditions (including the case when $\dot{x}_0 = 0$). If at the initial instant of time, t_0, the coordinates and the velocities of the particle are chosen in accordance with the equality

$$\frac{y_0}{x_0}=\frac{\dot{y}_0}{\dot{x}_0}, \tag{1.2.5}$$

then (1.2.4) leads to the following equation:

$$\frac{Y_0}{X_0}=\frac{y_0}{x_0}. \tag{1.2.6}$$

This means that at the initial moment of time, the force is directed along a straight line passing through the origin of coordinates, situated at the focus of the conic section. Since, in accordance with the statement of the problem, the unknown force must depend only on the position of a particle, and the position of the particle at the initial moment of time has been chosen arbitrarily, the force causing the motion of the particle along a conic section (1.2.2) must be central. Hence, the right-hand sides of Eqs. (1.2.1) can be represented in the form

$$X = xV(x, y), \qquad Y = yV(x, y), \tag{1.2.7}$$

where the function $V(x, y)$ is determined from Eq. (1.2.3) by taking into account the integral of the areas

$$x\dot{y} - \dot{x}y = c.$$

Finally, we obtain the following projections of the required force:

$$X=-\frac{c^2}{p}\frac{1}{r^2}\frac{x}{r}, \qquad Y=-\frac{c^2}{p}\frac{1}{r^2}\frac{y}{r}. \tag{1.2.8}$$

Hence, the required force is attractive with its centre at the focus of the conic section, and is inversely proportional

to the square of the distance between the moving particle and the centre.

Let us discuss the case when the conic section is central. The equation of the conic section can then be written in the form

$$Ax^2 + By^2 + 1 = 0, \tag{1.2.9}$$

assuming that the centre of the conic section has been chosen as the origin of rectangular coordinates.

Equation (1.2.9) immediately leads to

$$Ax\dot{x} + By\dot{y} = 0. \tag{1.2.10}$$

Taking into account the equations of motion (1.2.1), we get

$$A\dot{x}^2 + B\dot{y}^2 + AxX + ByY = 0. \tag{1.2.11}$$

From (1.2.10) and (1.2.11), we obtain the equation

$$\frac{\dot{x}}{y}(\dot{x}y - x\dot{y}) + \frac{x}{\dot{y}}(\dot{y}X - \dot{x}Y) = 0, \tag{1.2.12}$$

which holds at any instant of time and under any initial conditions (including the cases when $y = 0$, and $\dot{y} = 0$).

At the initial moment of time t_0, let us choose the coordinates and the velocities of a point in accordance with the equality

$$\frac{x_0}{y_0} = \frac{\dot{y}_0}{\dot{x}_0}.$$

Then, (1.2.12) leads to the equality

$$\frac{Y_0}{X_0} = \frac{y_0}{x_0}$$

and on the basis of the arguments made in the previous case, we come to the conclusion that the force is central and passes through the centre of the conic section. Hence

$$X = xV(x, y), \qquad Y = yV(x, y), \tag{1.2.13}$$

where the function V is determined from (1.2.11), and

$$V = A\dot{x}^2 + B\dot{y}^2. \tag{1.2.14}$$

Differentiating this expression with respect to time and taking into consideration Eq. (1.2.10), we find that $\dot{V} = 0$, and hence

$$X = cx, \qquad Y = cy \quad (c = \text{const}). \tag{1.2.15}$$

Thus, the required force is central and is directly proportional to the distance between the moving particle and the centre of the conic section.

1.3. Suslov's Problem. The next fundamental problem of dynamics is Suslov's problem [1] on finding a force function U which determines the forces causing the motion of a holonomic mechanical system with given integrals. This problem is a quite general inverse problem of dynamics of a mechanical system, based on the assumption that the force field is a potential field. Its solution can be also reduced to the construction of the right-hand sides of the equations of motion of a mechanical system. In Suslov's work [1], a statement of this problem has been given, and an analytical method has been proposed for solving this problem for a system with n degrees of freedom, from $n - 1$ independent integrals. With the help of a geometrical interpretation of this problem, Zhukovskii [6] has constructed the force function in an explicit form for mechanical systems with one or two degrees of freedom.

The problem of constructing a force function U for the case when a particle moves along a given plane curve

$$\omega(x, y) = 0 \tag{1.3.1}$$

is one of the simplest problems of this kind. We shall now solve this problem.

The set of unknown force functions U in the problem under investigation is determined from the integral of energy

$$U = T - h, \tag{1.3.2}$$

where the kinetic energy T is calculated for a unit mass after finding the projections of the velocity of a particle from the relation

$$\frac{\partial \omega}{\partial x} \dot{x} + \frac{\partial \omega}{\partial y} \dot{y} = 0. \tag{1.3.3}$$

In view of (1.3.1), this relation holds at any moment of time under any conditions and allows us to present the projection of the velocity of the particle as follows:

$$\begin{aligned} \dot{x} &= M(x, y)\frac{\partial\omega}{\partial y} + F_1(\omega, x, y), \\ \dot{y} &= -M(x, y)\frac{\partial\omega}{\partial x} + F_2(\omega, x, y), \end{aligned} \tag{1.3.4}$$

where $M(x, y)$, $F_1(\omega, x, y)$, and $F_2(\omega, x, y)$ are arbitrary functions, differentiable in the neighbourhood of the integral (1.3.1); besides, $F_1(0, x, y) = 0$, $F_2(0, x, y) = 0$.

Thus, the required set of force functions can be represented in the form

$$U = \frac{1}{2}M^2(x, y)\left[\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right] + \Phi(\omega, x, y) - h, \tag{1.3.5}$$

where $\Phi(\omega, x, y)$ is an arbitrary function, differentiable in the neighbourhood of $\omega(x, y)$ and satisfying the condition $\Phi(0, x, y) = 0$.

The solution of Bertrand's problem can be easily obtained from the relation (1.3.5) by assuming that the unknown force has already been proved to be central.

Indeed, for Bertrand's problem, we have

$$\omega \equiv \sqrt{x^2 + y^2} - ex = p$$

and hence

$$U = M^2(x, y)\frac{p}{r} - h.$$

Having determined M from the integral of the areas $x\dot{y} - \dot{x}y = c$, we obtain the force function

$$U = \frac{c^2}{p}\frac{1}{r} - h.$$

This gives the projections of the required force in the form (1.2.8).

1.4. Meshcherskii's Problem. Inverse problems of dynamics of a particle with varying mass, investigated by Meshcherskii [7], have proved to be of considerable interest both theoretically and from the application point of view. Here, the law of change in mass of the point and the velocity of

the escaping mass are required to be determined in such a way that a particle with varying mass should move along a given trajectory or under a given law in a given force field.

It should be noted that while in Bertrand's and Suslov's problems the equations of motion were sought for a mechanical system as a whole, the structure of these equations is supposed to be known in Meshcherskii's problem, and the parameters of the mechanical systems and the additional forces in the equations of motion are determined. The motion under these forces is one of the possible motions of the system under investigation, and this leads to the appropriate equations of motion of this system.

Thus, for example, in the problem of realizing the motion of a heavy particle with varying mass $m(t)$, in accordance with given laws of the change in the distance y and the height z

$$y = \varphi(t), \qquad z = \psi(t), \tag{1.4.1}$$

the equations of motion of the particle have the following form, defined by the problem itself:

$$\begin{aligned} m\ddot{y} &= \dot{m}(\mu - 1)\dot{y} - mf(z, v)\frac{\dot{y}}{v}, \\ m\ddot{z} &= \dot{m}(\eta - 1)\dot{z} - mf(z, v)\frac{\dot{z}}{y} - mg, \end{aligned} \tag{1.4.2}$$

where $f(z, v)$ is the drag per unit mass, $v = \sqrt{\dot{y}^2 + \dot{z}^2}$ is the velocity of the particle, $\mu = \mu(t)$ and $\eta = \eta(t)$ are the ratios of the projections of the velocities of the escaping mass and of the particle itself on the y- and z-axes.

The law of change in mass $(\dot{m})$ and the velocities of escaping mass (μ, η), admitting the given motion (1.4.1), are determined here from the conditions that the functions (1.4.1) form the particular solution of the differential equations (1.4.2). These conditions are obtained by substituting the law of motion (1.4.1) into Eqs. (1.4.2).

If strictly vertical and strictly horizontal motions ($\dot{\varphi} \not\equiv 0$, $\dot{\psi} \equiv 0$) are excluded from the problem in question, we

obtain the following conditions for determining the necessary laws of variation of the quantities μ, η, and m:

$$\mu = 1 + \frac{m}{\dot{m}}\left[\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{f(\psi(t),\ v_0)}{v_0}\right],$$

$$\eta = 1 + \frac{m}{\dot{m}}\left[\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} + \frac{f(\psi(t),\ v_0)}{v_0}\right], \tag{1.4.3}$$

$$v_0 = \sqrt{\dot{\varphi}^2 + \dot{\psi}^2}.$$

Thus, it is clear that the ultimate determination of the unknown quantities and hence the restoration of the equations of motion (1.4.2), admitting the given particular solution (1.4.1), is possible only under an additional condition. This condition can be imposed, for example, on the value of the relative velocity of the escaping mass

$$u_0 = \sqrt{(\mu - 1)^2 \dot{\varphi}^2 + (\eta - 1)^2 \dot{\psi}^2} \tag{1.4.4}$$

or on the angle between the directions of the velocities of the particle itself and escaping mass

$$\alpha = \arccos \frac{(\mu - 1)\dot{\varphi}^2 + (\eta - 1)\dot{\psi}^2}{u_0 v_0}. \tag{1.4.5}$$

1.5. Chaplygin-Goryachev Problem. In the development of the inverse problems of dynamics, a considerable importance has also been attached to the problem of determining such conditions on the mass geometry of a rigid body, and on the forces applied to this solid, under which the appropriate equations of motion of this body about a fixed point admit given integrals. Problems of this type have been investigated by a number of famous scientists. For instance, Chaplygin [8] has indicated the cases in which the equations of motion of a heavy rigid body with one fixed point have a linear particular integral with constant coefficients relative to the projections of the instantaneous angular velocity onto the principal axes. A more general problem is considered in Goryachev's work [9] where the given integrals are rational functions of first and second orders (the coefficients depend upon Euler's angles) relative to these projections, and we

seek the appropriate conditions imposed not only on the mass geometry, but also on the forces applied to the body.

Problems of this type are also reduced to the problems of constructing the equations of motion in accordance with given integrals, but differ from the inverse problems of dynamics considered above in that the very formulation of the problem completely defines a part of the equations (kinematic equations),

$$\begin{aligned}
\dot{\psi} &= \frac{x_1 \sin\varphi + x_2 \cos\varphi}{\sin\theta}, \\
\dot{\theta} &= x_1 \cos\varphi - x_2 \sin\varphi, \qquad (1.5.1) \\
\dot{\varphi} &= x_3 - (x_1 \sin\varphi + x_2 \cos\varphi) \cot\theta,
\end{aligned}$$

where x_1, x_2, x_3 are the projections of the instantaneous angular velocity of the rigid body onto the principal axes x, y, z of the ellipsoid of inertia constructed at a fixed point in the solid, and ψ, θ, φ are Euler's angles. The structure of the remaining equations (dynamic equations) is also well known:

$$\begin{aligned}
A\dot{x}_1 &= (B - C)\, x_2 x_3 + \mathscr{L}_1, \\
B\dot{x}_2 &= (C - A)\, x_3 x_1 + \mathscr{L}_2, \qquad (1.5.2) \\
C\dot{x}_3 &= (A - B)\, x_1 x_2 + \mathscr{L}_3.
\end{aligned}$$

The problem consists in finding the appropriate conditions, imposed on the moments of inertia A, B, C, relative to the principal axes, and on the projections $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ of the principal moments of the external forces on the same axes. Finally, as a result of the solution of the problem, the system of equations (1.5.1) and (1.5.2) becomes a closed system of differential equations, admitting given particular integrals.

By way of an example of solving the inverse problem of dynamics of a rigid body with one fixed point, let us determine the conditions of realization of regular precession in Lagrange's case [10].

In this case, the given particular integrals are the equations

$$\begin{aligned}
\omega_1 &\equiv x_1 \sin\varphi + x_2 \cos\varphi - n_1 \sin\theta_0 = 0, \\
\omega_2 &\equiv x_1 \cos\varphi - x_2 \sin\varphi = 0, \qquad (1.5.3) \\
\omega_3 &\equiv x_3 - n_1 \cos\theta_0 - n_2 = 0,
\end{aligned}$$

expressing the definition itself of regular precession ($\theta = \theta_0$, $\dot{\psi} = n_1$ and $\dot{\varphi} = n_2$ are constants).

The dynamic equations in Lagrange's case are written as follows:

$$\begin{aligned} A\dot{x}_1 &= (A - C)\, x_2 x_3 + Mgz_c x_5, \\ A\dot{x}_2 &= (C - A)\, x_3 x_1 - Mgz_c x_4, \\ \dot{x}_3 &= 0, \end{aligned} \tag{1.5.4}$$

where z_c is the distance between the centre of gravity and the fixed point, $x_4 = \sin\theta_0 \sin\varphi$, $x_5 = \sin\theta_0 \cos\varphi$, and Mg is the weight of the body.

The conditions that the relations (1.5.3) be the integrals of the system of equations (1.5.4) are obtained by differentiating these relations with respect to time and then substituting the values $\dot{x}_1$ and $\dot{x}_2$ from (1.5.4). These conditions have the form

$$\begin{aligned} Mgz_c x_4 &= (C - A)\, x_2 x_3 + An_1 n_2 \sin\theta_0 \cos\varphi, \\ Mgz_c x_5 &= (C - A)\, x_3 x_1 + An_1 n_2 \sin\theta_0 \sin\varphi. \end{aligned} \tag{1.5.5}$$

The required condition for the realization of the regular precession of a heavy rigid body with one fixed point in Lagrange's case is obtained by substituting into (1.5.5) the values of x_1, x_2, x_3, and x_4, x_5, expressed in terms of Euler's angles:

$$Mgz_c + (A - C)\, n_1^2 \cos\theta_0 = Cn_1 n_2. \tag{1.5.6}$$

Note that if $z_c = 0$, and $n_1 \neq 0$, we obtain the following equality from (1.5.6):

$$(A - C)\, n_1 \cos\theta_0 = Cn_2, \tag{1.5.7}$$

which is the necessary condition for the realization of the regular precession of a symmetrical rigid body during its rotation around a fixed point by inertia.

1.6. Poincaré-Cartan Problem. We conclude our review of classical inverse problems of dynamics by considering Cartan's problem [11] about the construction of equations of motion of mechanical systems, admitting a linear relative integral invariant. This problem consists in the following.

In the process of motion of a mechanical system, the Poincaré-Cartan integral along a closed curve C

$$J=\oint_C \sum_{i=1}^{n} p_i\,\delta q_i - H\,\delta t, \tag{1.6.1}$$

where H is the Hamiltonian of the system, and $q\ [q_1, \ldots, q_n]$ and $p\ [p_1, \ldots, p_n]$ are the vectors of the generalized velocities and momenta respectively, does not change its value upon an arbitrary displacement of this curve with a corresponding deformation along the tube of real trajectories of the representative point $M\ (q,\ p,\ t)$ in the extended phase space. It is required to construct the equations of motion of this mechanical system.

Let us suppose that the required equations of motion have the following form:

$$\begin{aligned} \dot{q}_i &= Q_i\,(q,\ p,\ t), \\ \dot{p}_i &= P_i\,(q,\ p,\ t) \quad (i=1,\ \ldots,\ n). \end{aligned} \tag{1.6.2}$$

We close the system of equations (1.6.2) by introducing the parameter β, so that

$$\frac{dt}{d\beta}=\tau\,(q,\ p,\ t), \tag{1.6.3}$$

where $\tau\,(q,\ p,\ t)$ is an arbitrary function.

The system of equations (1.6.2) and (1.6.3) defines a set of real trajectories of the representative point $M\ (q,\ p,\ t)$ in the extended phase space. Let us isolate from this set a tube of real trajectories whose directrix is some closed curve C, given by the parametric equations

$$q=q\,(\alpha), \qquad p=p\,(\alpha), \qquad t=t\,(\alpha), \tag{1.6.4}$$

where the parameter α changes from 0 to α_0, and

$$q\,(0)=q\,(\alpha_0), \qquad p\,(0)=p\,(\alpha_0), \qquad t\,(0)=t\,(\alpha_0). \tag{1.6.5}$$

Here, every point of the closed curve C defines a generatrix of the tube of real trajectories. Then, the coordinates of the representative point $M\ (q,\ p,\ t)$, which remains during

its motion on the tube of real trajectories isolated in this way, will depend on the parameters α and β:

$$q = q(\alpha, \beta), \quad p = p(\alpha, \beta), \quad t = t(\alpha, \beta). \quad (1.6.6)$$

Here, the change in α for β = const corresponds to the motion of the representative point along the closed curve surrounding the tube of trajectories, while the change in β for α = const corresponds to the motion of the representative point along one of the real trajectories.

Let us denote the differentiation with respect to the independent variable parameters α and β by δ and d, respectively, considering that these operations are commutative in view of their independence.

We replace q, p, and t in the integral J in (1.6.1) by their values in terms of the parameters α and β. Then, for a fixed value of β, the integral J (1.6.1) will be a curvilinear integral along some curve C surrounding the tube of real trajectories. As this integral is an invariant of a mechanical system, we get

$$dJ = \oint_C \sum_{i=1}^{n} (dp_i\, \delta q_i + p_i\, d\,\delta q_i) - dH\,\delta t - H\, d\,\delta t = 0. \quad (1.6.7)$$

Considering that $d\,\delta q = \delta\, dq$, $d\,\delta t = \delta\, dt$ and integrating by parts, we obtain

$$\oint_C \sum_{i=1}^{n} (dp_i\, \delta q_i - dq_i\, \delta p_i) - dH\,\delta t + \delta H\, dt = 0, \quad (1.6.8)$$

and

$$\oint_C \sum_{i=1}^{n} \left[\left(dp_i + \frac{\partial H}{\partial q_i} dt\right) \delta q_i + \left(-dq_i + \frac{\partial H}{\partial p_i} dt\right) \delta p_i \right] + \left(-dH + \frac{\partial H}{\partial t} dt\right) \delta t = 0. \quad (1.6.9)$$

Considering that $d\beta = dt/\tau(q, p, t)$ and using the equations of motion (1.6.2) assumed above, we obtain

$$\oint_C \left\{ \sum_{i=1}^{n} \left[\left(P_i + \frac{\partial H}{\partial q_i}\right) \delta q_i + \left(-Q_i + \frac{\partial H}{\partial p_i}\right) \delta p_i \right] + \left(-\frac{dH}{dt} + \frac{\partial H}{\partial t}\right) \delta t \right\} \tau(q, p, t) = 0. \quad (1.6.10)$$

Further, in view of the arbitrary nature of the function $\tau(q, p, t)$, and the independence of δq, δp, and δt, we get

$$Q_i = \frac{\partial H}{\partial p_i}, \quad P_i = -\frac{\partial H}{\partial q_i}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (i = 1, \ldots, n). \tag{1.6.11}$$

Therefore, if a mechanical system has the Poincaré-Cartan integral invariant

$$J = \oint_C \sum_{i=1}^{n} p_i \, \delta q_i - H(q, p, t) \, \delta t, \tag{1.6.1}$$

in the extended phase space $\{q, p, t\}$ the motion of this system is described by the canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (i = 1, \ldots, n) \tag{1.6.12}$$

with the Hamiltonian $H(q, p, t)$ appearing in the integral invariant.

Note that the problem of constructing the equations of motion is also stated and solved for an invariant Poincaré integral

$$\oint_C \sum_{i=1}^{n} p_i \delta q_i$$

taken along the arbitrary closed curve C of simultaneous states of a mechanical system for an arbitrary displacement of this curve with a corresponding deformation along the tube of real trajectories of the representative point $M(q, p)$ in the phase space.

Here, the motion of a mechanical system can also be described by the canonical equations (1.6.12) but the corresponding Hamiltonian remains indefinite within the range of the formulated problem.

To conclude this review, let us mention that the inverse problems investigated here are of a fundamental nature in that the solutions of these problems with a subsequent generalization of their physical interpretation have opened new perspectives and phenomena in natural sciences. Some of these have proved to be the starting points in the for-

mulation and development of modern branches of science on the controlled motion of material systems.

Thus, for example, the solution of Newton's problem has resulted in the discovery of one of the most remarkable laws of nature—the law of universal gravitation, while the solution of Cartan's problem has led to the establishment of a new principle in analytical dynamics—the principle of conservation of momentum and energy. A mathematical generalization of the investigated problems of the dynamics of a rigid body with one fixed point and the dynamics of a particle with varying mass, as well as the problems of the type of Suslov's problem, have nowadays become the basic problems of space science, rocket dynamics, and the theory of construction of programmed motion systems.

Sec. 2. Statement of the Inverse Problems of Dynamics

The concept of the inverse problems of dynamics, generalizing the classical problems considered above, can be stated as follows:

The problems of determination of active forces and moments applied to a mechanical system, the determination of the system parameters and constraints, additionally imposed on it and resulting in the motion with given properties as one of the possible motions of the mechanical system in question, are called the inverse problems of dynamics. Here, the properties of motion can be described in quite different ways. For example, these could be in the form of quantitative and qualitative restrictions on the coordinates and velocities of motion, in the form of the integral invariants of the systems, and so on.

Let us suppose that the properties of motion of a mechanical system, whose state is defined by the vector of generalized coordinates $y\ [y_1, \ldots, y_n]$, and the vector of generalized velocities $\dot{y}\ [\dot{y}_1, \ldots, \dot{y}_n]$ are given in the form of the manifold

$$\Omega:\ \omega_\mu (y, \dot{y}, t) = c_\mu \qquad (\mu = 1, \ldots, m \leqslant n), \quad (2.1)$$

where some of the constants c_μ may be equal to zero.

As regards the functions ω_μ, we assume that $\omega_\mu (y, \dot{y}, t) \in$

$\in C'$, the equalities $\omega_\mu(y, \dot{y}, t) = c_\mu$ are compatible and independent in some domain of the phase space $G\{y, \dot{y}\}$ for $t \geqslant t_0$.

The given manifold Ω of the properties of motion is essentially an integral manifold of the corresponding equations of motion of the mechanical system in question. Therefore, it is natural that in order to solve an inverse problem of dynamics, we must construct the equations of motion of the mechanical system under investigation in accordance with the given integral manifold Ω in such a way that the expressions $\omega_\mu(y, \dot{y}, t) = c_\mu$ are the first $(c_\mu \neq 0)$ or particular $(c_\mu = 0)$ integrals of these equations. From the equations thus constructed, we must determine the unknown generalized forces, parameters, and constraints admitting the motion of the system with given properties (2.1).

In particular cases, when the structure of equations of motion is known, while additional forces and the parameters of the mechanical system in question, required for imparting the given properties to the motion of the system, are unknown, it is necessary to determine the equations of motion in accordance with the given integral manifold, and to determine the unknown quantities from the equations thus restored.

In those cases when only a part of the equations of the mechanical system under investigation is known, the missing equations are to be constructed according to the given integral manifold in order to solve the inverse problems, and from the system thus closed we determine the unknown generalized forces, parameters and constraints, which make the motion with the given properties possible.

Thus, the solution of the inverse problems of dynamics in a quite general mathematical interpretation is reduced to the construction of the equations of motion of mechanical systems in accordance with the integral manifold given in the form of properties of their motion. If we confine ourselves to mechanical systems whose motion is described by ordinary differential equations, the appropriate problems of the construction of equations can be formulated in one of the following ways depending on their importance from the application point of view.

1. The fundamental problem of constructing the equations of motion. *Construct the system of equations*

$$\ddot{y}_\nu = Y_\nu (y, \dot{y}, t) \quad (\nu = 1, \ldots, n) \tag{2.2}$$

in accordance with the given integral manifold

$$\Omega: \omega_\mu (y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant n). \tag{2.3}$$

2. The restoration of the equations of motion. *Given a system of equations*

$$\ddot{y}_\nu = Y_{0\nu} (y, \dot{y}, v, t) \quad (\nu = 1, \ldots, n). \tag{2.4}$$

Determine the vector function $v\,[v_1 (y, \dot{y}, t), \ldots, v_k (y, \dot{y}, t)]$ *in accordance with the given integral manifold*

$$\Omega: \omega_\mu (y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant n). \tag{2.5}$$

3. Closure of the equations of motion. *Given a system of equations*

$$\ddot{y}_\nu = Y_{0\nu} (y, \dot{y}, u, \dot{u}, t) \quad (\nu = 1, \ldots, n). \tag{2.6}$$

Construct a system of closing equations

$$\ddot{u}_\rho = U_\rho (y, \dot{y}, u, \dot{u}, t) \quad (\rho = 1, \ldots, r) \tag{2.7}$$

in accordance with the given integral manifold

$$\Omega: \omega_\mu (y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant r). \tag{2.8}$$

These problems are formulated under the assumption that the unknown functions Y_ν, $v_\varkappa$, U_ρ ($\nu = 1, \ldots, n$; $\varkappa = 1, \ldots, k$; $\rho = 1, \ldots, r$) belong to the class of functions such that the solution of corresponding equations exists and is unique in some ε-neighbourhood Ω_ε of a given manifold Ω. Moreover, if the representative point turns out to be on the integral manifold Ω at the initial moment of time, it will continue to move along this manifold.

Naturally, the above formulations of the problems of constructing the equations of motion are not the only possible versions. Thus, for example, it may so happen that the given properties of motion depend only on a part of the variables y, $\dot{y}$, and that these properties are only geometrical or only kinematic. The properties of motion may be partly

given in the form of inequalities. Some equations to be constructed may happen to be first-order differential equations, and so on.

However, the above basic versions of the formulation of these problems include all the possible modifications of the problems of constructing the differential equations, from the point of view of their application as well as in the technique of solving them, because these versions are either generalizations or particular cases of the possible modifications of the problems of constructing the equations of motion.

It should be noted that the above versions of the statement of the inverse problems of constructing differential equations are essentially mathematical generalizations of some well-known inverse problems of dynamics. In fact, the basic problem of constructing the equations of motion can be regarded as a generalization of the inverse problems of dynamics in which it is necessary to determine the generalized forces under the action of which the motion of a mechanical system with given properties is possible. In particular, this could mean a further generalization of Newton's, Bertrand's, Suslov's, and Cartan's problems. The problems of restoration of equations of motion are a mathematical formulation of all the inverse problems of dynamics, where the values of parameters of a mechanical system and additional forces are required to be determined in accordance with the given properties of motion of this system as, for example, in Meshcherskii's problem. The problem of closure of given equations of motion embraces problems of dynamics, where a part of equations describing the motion of some particles in a mechanical system is given, and the equations of motion of other particles in the system are required to be found from the known properties of motion of the entire system. It can also cover problems where the kinematic equations of motion of a mechanical system are given, and the dynamic equations of motion of this system are required to be found as, for example, in Goryachev's problem.

Sec. 3. Solution of the Inverse Problems of Dynamics

It follows from the above discussion that the solution of the inverse problems of dynamics is reduced to the solution of the inverse problems in the theory of ordinary differential

equations, where the latter are constructed or completed in accordance with the given first or particular integrals.

The inverse problem in the theory of differential equations in the form of a problem of constructing a set of systems of equations in accordance with the given particular integrals was first formulated by Erugin in [12], where the technique of solving this problem has also been described. In [13-22], the problems of the construction of differential equations have been stated and solved in connection with the inverse problems of dynamics and various problems on controlled motions of material systems, and in particular, in connection with the problems of the analytical construction of systems of programmed motion. It was found that Erugin's method, proposed for the construction of differential equations in accordance with given particular integrals, permits not only the construction of the equations of motion of a mechanical system in accordance with the given properties of one of the possible motions of this system, but also the construction of these equations with regard to additional requirements, for instance, the requirement of stability, or the optimality of the given motion.

In order to solve the inverse problems of dynamics by using this technique, we first formulate the necessary and sufficient conditions requiring that the given integrals really form an integral manifold of the system of differential equations being constructed. These conditions are obtained by equating the derivatives of the given integrals which, in view of the required equations, are formed of arbitrary functions, vanishing at the given integral manifold.

The equations obtained here will be the necessary conditions for the realization of the given motion for the mechanical system in question; necessary, because in addition to these conditions, the realization of the motion of a system with given properties also requires that the initial state of the system have the given properties [22].

In the case of the basic problem of constructing the equations of motion, the necessary conditions for the realization of a motion with the given properties (2.3) are written in the following form [12, 22]:

$$(\operatorname{grad}_{\dot{y}} \omega_\mu \cdot Y) + (\operatorname{grad}_{y} \omega_\mu \cdot \dot{y}) + \frac{\partial \omega_\mu}{\partial t} = \Phi_\mu(\omega, y, \dot{y}, t) \quad (\mu = 1, \ldots, m), \tag{3.1}$$

where $Y\,[Y_1, \ldots, Y_n]$ is the vector function of the right-hand sides of Eqs. (2.2), $\Phi_\mu(\omega, y, \dot{y}, t)$ are arbitrary functions for $c_\mu = 0$, vanishing on the integral manifold (2.3), and $\Phi_\mu = 0$ for $c_\mu \neq 0$.

These conditions are considered as the initial equations in functions Y_ν ($\nu = 1, \ldots, n$), necessary for constructing the appropriate equations of motion and for subsequently determining the unknown generalized forces, under the action of which the motion of a mechanical system with given properties is one of the possible motions of the mechanical system in question.

In the problem of restoration, the necessary conditions for the realization of a motion with the given properties (2.5) have the same form as in the basic problem. From the conditions (3.1), we can determine the functions Y_ν ($\nu = 1, \ldots, n$) which, according to the statement of the problem itself, have to be identical with the right-hand sides of the equations of motion (2.4), and hence

$$Y_{0\nu}(y, \dot{y}, v, t) = Y_\nu \quad (\nu = 1, \ldots, n). \tag{3.2}$$

The equations obtained here form a system of equations in functions $v_\varkappa(y, \dot{y}, t)$ ($\varkappa = 1, \ldots, k$), which ultimately determine the unknown additional forces and parameters of the mechanical system under investigation [22].

It should be noted that the ultimate determination of the functions $v_\varkappa(y, \dot{y}, t)$ ($\varkappa = 1, \ldots, k$) from Eqs. (3.2) is possible only when the equations (3.2) form a system of compatible equations, a fact which must be taken into account while stating the problem itself in specific cases, and while choosing the arbitrary functions $\Phi_\mu(\omega, y, \dot{y}, t)$ in the process of forming the conditions (3.1).

While solving the closure problem, the necessary conditions for the realization of the motion with the given properties (2.8) will have the following form [22]:

$$(\operatorname{grad}_{\dot{u}} \dot{\omega}_\mu \cdot U) + (\operatorname{grad}_{\dot{y}} \dot{\omega}_\mu \cdot Y_0) + (\operatorname{grad}_{y} \dot{\omega}_\mu \cdot \dot{y}) + (\operatorname{grad}_{u} \dot{\omega}_\mu \cdot \dot{u})$$
$$+ \frac{\partial \dot{\omega}_\mu}{\partial t} = \Phi_\mu(\omega, y, \dot{y}, t) \quad (\mu = 1, \ldots, m), \tag{3.3}$$

where $Y_0\,[Y_{01}, \ldots, Y_{0n}]$, $U\,[U_1, \ldots, U_r]$ are the vector functions of the right-hand sides of Eqs. (2.6) and (2.7);

$$\dot{\omega}_\mu = (\operatorname*{grad}_{\dot{y}} \omega_\mu \cdot Y_0) + (\operatorname*{grad}_{y} \omega_\mu \cdot \dot{y}) + \frac{\partial \omega_\mu}{\partial t}.$$

The conditions (3.3) are the initial equations for finding the functions U_ρ $(\rho = 1, \ldots, r)$ required for the construction of the unknown closing equations (2.7).

Note that while solving the inverse problems of dynamics, the unknown system of equations of motion, having the same prescribed integral manifold (2.1), can be represented in various forms [15]. All this depends on the form in which the vector functions Y, U of the right-hand sides of these equations are sought while solving the appropriate system (indefinite for $m < n, r$) of linear equations (3.1) and (3.3).

Note also that in the general case, each inverse problem of dynamics has a multiple-valued solution. This can be explained in the first place by the fact that Eqs. (3.1) for $m < n$, and (3.3) for $m < r$ do not define uniquely the unknown vector functions Y and U. Secondly, these equations themselves contain arbitrary functions $\Phi_\mu(\omega, y, \dot{y}, t)$ if zero c_μ's are present. Naturally, the conditions imposed above that these functions should vanish on the integral manifold Ω and the conditions of existence and uniqueness of the solution in the domain Ω_ε of the equations of motion being constructed do not remove this ambiguity. All this allows us to solve the inverse problems of dynamics in combination with the problems of stability and optimization of the given motion, and on the whole in conjunction with the additional requirements about the dynamical indices of motion of the mechanical system in question.

Sec. 4. Construction of the Equations of Motion

We shall now solve the above-mentioned basic problem of constructing the equations of motion of a mechanical system

$$\ddot{y}_\nu = Y_\nu(y, \dot{y}, t) \quad (\nu = 1, \ldots, n) \tag{4.1}$$

in such a way that the motion with the following properties is one of the possible motions of this system:

$$\Omega: \omega_\mu (y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant n). \tag{4.2}$$

It has been shown that the solution of this problem is reduced to the construction of differential equations (4.1) in accordance with the given integral manifold Ω (4.2).

The problem is solved by assuming that the given integrals (4.2) are compatible and independent, the functions $\omega_\mu (y, \dot{y}, t)$ are bounded, continuous, and differentiable with respect to all variables in some domain $G\{y, \dot{y}\}$ of the phase space for any $t \geqslant t_0$, and the right-hand sides of equations (4.1) are sought in the class of functions admitting the existence and uniqueness of the solution in some ε-neighbourhood Ω_ε of the integral manifold Ω. Here, the initial conditions $(y, \dot{y})|_{t=t_0} \in \Omega$ for $t > t_0$ will lead to the motion of the representative point $M(y, \dot{y})$ along the given integral manifold Ω.

In the problem under investigation, the conditions for realizing the motion with the given properties (4.2) are written in the form of the following equalities:

$$(\operatorname{grad}_{\dot{y}} \omega_\mu \cdot Y) = \Phi_\mu (\omega, y, \dot{y}, t) - \varphi_\mu \quad (\mu = 1, \ldots, m), \tag{4.3}$$

where

$$\varphi_\mu = (\operatorname{grad}_y \omega_\mu \cdot \dot{y}) + \frac{\partial \omega_\mu}{\partial t},$$

$\Phi_\mu (\omega, y, \dot{y}, t)$ is a function identically equal to zero for $c_\mu \neq 0$, arbitrary for $c_\mu = 0$, and vanishing on the integral manifold Ω. Such a function, for instance, can be the holomorphic function of variables $\omega_1, \ldots, \omega_m$ in the domain Ω_ε for $t \geqslant t_0$, the expansion of which in powers of these variables begins with terms of not lower than the first order.

The equalities (4.3) serve as the equations for determining the right-hand sides Y_ν of the required equations (4.1).

For the case when $m = n$, a direct solution of Eqs. (4.3) shows that the required equations form the following system:

$$\ddot{y}_\nu = \sum_{i=1}^{n} \frac{\Delta^{i\nu}}{\Delta} (\Phi_i - \varphi_i), \tag{4.4}$$

where

$$\Delta = \det \left\| \frac{\partial \omega}{\partial \dot{y}} \right\|_m^m \neq 0,$$

and $\Delta^{i\nu}$ is the cofactor of the (i, ν)-th element of the determinant Δ.

If $m < n$, it is more convenient to seek the vector function Y of the right-hand sides of equations in the form of the following sum [23]:

$$Y = Y^\nu + Y^\tau, \tag{4.5}$$

where the vector Y^ν is orthogonal to the manifold

$$\Omega_{\dot{y}} \{\omega (y, \dot{y}, t)_{y=\mathrm{inv}} = 0\} \tag{4.6}$$

and is determined to within Lagrangian multipliers λ_i

$$Y^\nu = \sum_{i=1}^{m} \lambda_i \operatorname{grad}_{\dot{y}} \omega_i, \tag{4.7}$$

while the vector Y^τ is a component of the vector function along the manifold $\Omega_{\dot{y}}$ (4.6) and is determined by the condition

$$(\operatorname{grad}_{\dot{y}} \omega_\mu \cdot Y^\tau) = 0 \quad (\mu = 1, \ldots, m). \tag{4.8}$$

After substituting the vector function Y in the form of the sum (4.5) into the conditions for realization of the motion (4.3), we obtain

$$(\operatorname{grad}_{\dot{y}} \omega_\mu \cdot Y^\nu) = \Phi_\mu - \varphi_\mu. \tag{4.9}$$

Taking into account the value of Y^ν in the form (4.7), we get

$$\lambda_i = \frac{1}{\Gamma} \sum_{j=1}^{m} \Gamma_{ij} (\Phi_j - \varphi_j) \quad (i = 1, \ldots, m),$$

where

$$\Gamma = \det \| \operatorname{grad}_{\dot{y}} \omega_i \cdot \operatorname{grad}_{\dot{y}} \omega_j \|_m^m \neq 0,$$

and Γ_{ij} is a cofactor of the (i, j)-th element in the determinant Γ. Thus, we have

$$Y^{\nu} = \frac{1}{\Gamma} \sum_{i, j}^{1, m} \Gamma_{ij} (\Phi_j - \varphi_j) \operatorname{grad}_{\dot{y}} \omega_i. \tag{4.10}$$

The components of the vector function Y^{τ} are determined by solving an indefinite system of linear equations (4.8), and can be represented in the form

$$\begin{aligned} Y_r^{\tau} &= -\sum_{s=m+1}^{n} D^{rs} Q_s \quad (r = 1, \ldots, m), \\ Y_s^{\tau} &= D Q_s \quad (s = m+1, \ldots, n), \end{aligned} \tag{4.11}$$

where

$$D = \det \left\| \frac{\partial \omega}{\partial \dot{y}} \right\|_m^m,$$

and D^{rs} is the determinant obtained from D by replacing ts r-th column by the s-th column of the matrix $\left\| \frac{\partial \omega}{\partial \dot{y}} \right\|_n^m$, and $Q_s = Q_s (y, \dot{y}, t)$ are arbitrary functions.

Thus, the required system of equations (4.1) can be represented in the form

$$\begin{gathered} \ddot{y}_r = \frac{1}{\Gamma} \sum_{i, j}^{1, m} \Gamma_{ij} (\Phi_j - \varphi_j) \frac{\partial \omega_i}{\partial \dot{y}_r} - \sum_{s=m+1}^{n} D^{rs} Q_s \\ (r = 1, \ldots, m), \\ \ddot{y}_s = \frac{1}{\Gamma} \sum_{i, j}^{1, m} \Gamma_{ij} (\Phi_j - \varphi_j) \frac{\partial \omega_i}{\partial \dot{y}_s} + D Q_s \\ (s = m+1, \ldots, n). \end{gathered} \tag{4.12}$$

It is apparent that the solution of the general problem of constructing the equations of motion contains the functions $\Phi_j(\omega, y, \dot{y}, t)$ (for $c_j = 0$) and $Q_s(y, \dot{y}, t)$ (for $m < n$), which are indeterminate in the framework of the problem in question. Naturally, these functions must be chosen in such a way that the conditions of the existence and uniqueness of the solution of the system of equations (4.12) are satisfied in the domain Ω_ε.

The system (4.12) of the equations of motion admitting the motion with the given properties (4.2) can be represented in the form of the following vector equation:

$$\ddot{y} = \frac{1}{\Gamma} \sum_{i,j}^{1,m} \Gamma_{ij} (\Phi_j - \varphi_j) \operatorname{grad}_{\dot{y}} \omega_i + Y^\tau, \tag{4.13}$$

where the vector Y^τ is defined by the conditions (4.8).

The solution obtained here can be used for constructing the equations of motion in a number of inverse problems of dynamics on account of the scope of the formulation of the solved problem, as well as the universality of the technique applied while solving it.

It should be noted that in some particular cases while solving the inverse problems of dynamics, it is expedient to construct the equations of motion by using only some of the given integrals first, and then completing the equations by utilizing the remaining given integrals.

Sec. 5. Kepler's Planetary Motion

Let us consider the solution of Newton's problem by using the procedure described above.

We write the properties of motion of a planet in the form

$$\Omega: \begin{cases} \omega_1 \equiv r - ex = p \quad (r = \sqrt{x^2 + y^2}), & (5.1) \\ \omega_2 \equiv \dot{x}y - x\dot{y} = c. & (5.2) \end{cases}$$

These properties are the first integrals of the equations of motion of the planet

$$\ddot{x} = X(x, y), \qquad \ddot{y} = Y(x, y). \tag{5.3}$$

First of all, we construct the equations of motion by using only the area integral $\omega_2 = c$.

The necessary and sufficient condition for the equations of motion (5.3) of a planet to have this integral is written in the form

$$\frac{\partial \omega_2}{\partial \dot{x}} X + \frac{\partial \omega_2}{\partial \dot{y}} Y + \frac{\partial \omega_2}{\partial x} \dot{x} + \frac{\partial \omega_2}{\partial y} \dot{y} = 0,$$

or

$$xY - yX = 0. \tag{5.4}$$

The right-hand sides of Eqs. (5.3) can be immediately found from this equation:

$$X = xV(x, y), \qquad Y = yV(x, y). \tag{5.5}$$

For the time being, these contain the arbitrary function $V(x, y)$.

In order to determine this function, we use the equation of the trajectory of the planet, $\omega_1 = p$. We have

$$\ddot{\omega}_1 \equiv \frac{c^2}{r^3} + \frac{x}{r} X + \frac{y}{r} Y - eX = 0. \tag{5.6}$$

This leads to the required function

$$V(x, y) = -\frac{c^2}{p} \frac{1}{r^3} \tag{5.7}$$

and the final form of the equations of motion of a planet with mass m

$$\begin{aligned} m\ddot{x} &= -\frac{mc^2}{p} \frac{x}{r^3}, \\ m\ddot{y} &= -\frac{mc^2}{p} \frac{y}{r^3}. \end{aligned} \tag{5.8}$$

The right-hand sides of these equations are the projections of the force causing the motion of the planet with a constant sector velocity along an ellipse having the Sun at one of its foci. Further, using the property of motion of planets concerning the time of their revolution around the Sun, we finally arrive at the well-known solution of Newton's problem.

Sec. 6. Restoration of Equations of Motion

In the problems on restoration of equations of motion of mechanical systems, the structure of the equations

$$\ddot{y}_\nu = Y_{0\nu}(y, \dot{y}, v, t) \quad (\nu = 1, \ldots, n) \tag{6.1}$$

is supposed to be known, and it is required to determine the vector function $v\,[v_1, \ldots, v_k]$ of the system parameters and the forces additionally applied to the system in such a way that the motion of a mechanical system with the given properties

$$\Omega\colon\ \omega_\mu(y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant n) \tag{6.2}$$

should be one of its possible motions.

Under the above assumptions about the given ω_μ and the required function $v_\varkappa$, the problem of restoration of the equations of motion is solved in the following order [22].

1. The system of differential equations, for which the given manifold Ω (6.2) is integral, is composed. This system has been constructed earlier and is represented by the expression (4.13).

2. The right-hand sides of these equations are equated to the right-hand sides of the given equation (6.1). This leads to the following equalities:

$$\frac{1}{\Gamma}\sum_{i,j}^{1,m}\Gamma_{ij}\left[\Phi_j - (\operatorname{grad}_y \omega_j \cdot \dot{y}) - \frac{\partial \omega_j}{\partial t}\right]\frac{\partial \omega_i}{\partial \dot{y}_\nu} + Y_\nu^\tau$$
$$= Y_{0\nu}(y, \dot{y}, v, t) \quad (\nu = 1, \ldots, n). \tag{6.3}$$

3. The required functions $v_\varkappa$ $(\varkappa = 1, \ldots, k)$ are determined from Eqs. (6.3).

It should be noted that the solvability of Eqs. (6.3) in the functions $v_\varkappa$ $(\varkappa = 1, \ldots, k)$ is necessary for an ultimate solution of the inverse problem of dynamics under investigation. This requirement will result in the imposition of additional conditions on the functions Φ_j (for $c_j = 0$) and $Y_{m+1}^\tau, \ldots, Y_n^\tau$ appearing in Eqs. (6.3), as well as on the relations of the preset numbers n, m, and k.

Sec. 7. Self-excited Gyroscope

By way of an example, let us consider the problem of restoration of the equations of motion for a self-excited gyroscope [24]

$$\begin{aligned} A\dot{x}_1 &= (B - C)\, x_2 x_3 + \mathcal{L}_1, \\ B\dot{x}_2 &= (C - A)\, x_3 x_1 + \mathcal{L}_2, \\ C\dot{x}_3 &= (A - B)\, x_1 x_2 + \mathcal{L}_3, \end{aligned} \tag{7.1}$$

where A, B, and C are the principal moments of inertia of the gyroscope, $\omega\,[x_1,\ x_2,\ x_3]$ is the vector of its instantaneous angular velocity, and $\mathcal{L}\,[\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3]$ is the moment of the forces about a fixed point, depending on the projections x_1, x_2, x_3 of the instantaneous angular velocity ω onto the principal axes.

We consider the following problem:

Define the damping moment $\mathcal{L}$ so that the motion of a gyroscope has the following given properties:

$$\Omega: \begin{cases} \omega_1 \equiv e^{2\lambda t} T = \text{const}, \\ \omega_2 \equiv e^{2\lambda t} G^2 = \text{const}, \end{cases} \tag{7.2}$$

where $T = \frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_3^2)$ is the kinetic energy of the gyroscope, $G^2 = A^2x_1^2 + B^2x_2^2 + C^2x_3^2$ is the square of the kinetic moment calculated about fixed point, and λ is a positive constant.

In order to solve this problem, let us first construct the conditions for realizing the motion with properties Ω (7.2):

$$\begin{aligned} x_1\mathcal{L}_1 + x_2\mathcal{L}_2 + x_3\mathcal{L}_3 &= -2\lambda T, \\ Ax_1\mathcal{L}_1 + Bx_2\mathcal{L}_2 + Cx_3\mathcal{L}_3 &= -\lambda G^2. \end{aligned} \tag{7.3}$$

After this, we determine the projections $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ of the required moment $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}_1 &= \frac{-\lambda\,(2BT - G^2) + \mathcal{L}_3\,(C - B)\, x_3}{x_1\,(B - A)}, \\ \mathcal{L}_2 &= \frac{-\lambda\,(G^2 - 2AT) + \mathcal{L}_3\,(A - C)\, x_3}{x_2\,(B - A)}, \end{aligned} \tag{7.4}$$

where $\mathcal{L}_3$ is an arbitrary function of x_1, x_2, x_3.

It is clear that the solution of the problem contains one arbitrary function $\mathcal{L}_3$, which may be determined from some additional requirements.

For instance, we require that the damping be caused with the help of the viscous forces, and that $\mathcal{L}_3 = -\lambda C x_3$. We shall then get other projections also of the required damping moment

$$\mathcal{L}_1 = -\lambda A x_1, \qquad \mathcal{L}_2 = -\lambda B x_2, \tag{7.5}$$

and consequently, the equations of motion of the gyroscope

$$\begin{aligned} A\dot{x}_1 &= (B - C)\, x_2 x_3 - \lambda A x_1, \\ B\dot{x}_2 &= (C - A)\, x_3 x_1 - \lambda B x_2, \\ C\dot{x}_3 &= (A - B)\, x_1 x_2 - \lambda C x_3, \end{aligned} \tag{7.6}$$

constructed fully in accordance with the given properties of its motion.

Sec. 8. Closure of Equations of Motion

In the inverse problems of dynamics, whose solution results in the closure of the equations of motion, a part of the equations of motion (for example, the equations of motion of the main object of the system)

$$\ddot{y}_\nu = Y_{0\nu}(y, y, u, \dot{u}, t) \quad (\nu = 1, \ldots, n) \tag{8.1}$$

is known a priori; it is required to construct the necessary number of equations (for instance, the equation describing the motion of the auxiliary equipment)

$$\ddot{u}_\rho = U_\rho(y, \ddot{y}, u, \dot{u}, t) \quad (\rho = 1, \ldots, r) \tag{8.2}$$

in such a way that the system of equations (8.1) and (8.2) should be a closed system and that the given manifold

$$\Omega\colon\ \omega_\mu(y, \dot{y}, t) = c_\mu \quad (\mu = 1, \ldots, m \leqslant r) \tag{8.3}$$

should be an integral manifold for this system [22].

Problems of this type are formulated under the assumption that the functions $\omega_\mu(y, \dot{y}, t)$ satisfy, in addition to the

conditions mentioned above, the additional requirement

$$\dot{\omega}_\mu \equiv \left\{ (\operatorname{grad}_{\dot{y}} \omega_\mu \cdot Y_0) + (\operatorname{grad}_{y} \omega_\mu \cdot \dot{y}) + \frac{\partial \omega_\mu}{\partial t} \right\} \in C'$$

in some domain $G\{y, \dot{y}, u, \dot{u}\}$ of the phase space for $t \geqslant t_0$.

First of all, in order to solve the problem of closure of equations of motion under investigation, we construct the appropriate equations describing the conditions for realization of the motion (3.3):

$$(\operatorname{grad}_{\dot{u}} \dot{\omega}_\mu \cdot U) = \Phi_\mu (\omega, y, \dot{y}, t) - \varphi_\mu^* \quad (\mu = 1, \ldots, m), \tag{8.4}$$

where

$$\varphi_\mu^* = (\operatorname{grad}_{\dot{y}} \dot{\omega}_\mu \cdot Y_0) + (\operatorname{grad}_{y} \dot{\omega}_\mu \cdot \dot{y}) + (\operatorname{grad}_{u} \dot{\omega}_\mu \cdot \dot{u}) + \frac{\partial \dot{\omega}_\mu}{\partial t}.$$

After this, the right-hand sides of the required equations (8.2) are determined from these conditions with the help of the methods described above.

Finally, the closed system of equations will have the following form in vector notation:

$$\begin{aligned} \ddot{u} &= \frac{1}{\Gamma^*} \sum_{i, j}^{i, m} \Gamma_{ij}^* (\Phi_j - \varphi_j^*) \operatorname{grad}_{\dot{u}} \dot{\omega}_i + U^\tau, \\ \ddot{y} &= Y_0 (y, \dot{y}, u, \dot{u}, t), \end{aligned} \tag{8.5}$$

where $\Gamma^* = \det \| \operatorname{grad}_{\dot{u}} \dot{\omega}_i \cdot \operatorname{grad}_{\dot{u}} \dot{\omega}_j \|_m^m \neq 0$, Γ_{ij}^* is the cofactor of the (i, j)-th element of the determinant Γ^*, $\Phi_j = \Phi_j (\omega, y, \dot{y}, t)$ is a function identically equal to zero for $c_j \neq 0$, arbitrary for $c_j = 0$, and equal to zero for $\omega = 0$, and U^τ is a vector function satisfying the conditions $(\operatorname{grad}_{\dot{u}} \dot{\omega}_\mu \cdot U^\tau) = 0 \quad (\mu = 1, \ldots, m)$.

Obviously, the solution of the problem of closure of equations of motion also includes the functions Φ_j (for $c_\mu = 0$) and U_ρ^τ (for $r < m$), which are indeterminate in the framework of the problem formulated above.

Sec. 9. Differential Servosystems

As an example of closure of equations of motion, let us consider one of the problems of differential servomotion of two particles one of which follows the motion of the other [25].

Suppose that we are given the equation of motion of a particle M_0 of mass m_0:

$$m_0 \ddot{\mathbf{r}}_0 = \mathbf{F}_0, \tag{9.1}$$

where $\mathbf{F}_0 = \mathbf{F}_0 (\mathbf{r} - \mathbf{r}_0, \mathbf{v} - \mathbf{v}_0)$ *is the force applied to the particle* M_0; $\mathbf{r}_0$, $\mathbf{r}$, $\mathbf{v}_0$, and $\mathbf{v}$ *are the radius vectors and velocities of the particles M_0 and M respectively. It is required to determine the force* $\mathbf{F}$ *applied to the particle M with mass m, so that the motions of the particles with properties*

$$\Omega: \begin{cases} \omega_1 \equiv |\mathbf{r}_0 - \mathbf{r}| = l - \mathrm{const}, \\ \omega_2 \equiv ((\mathbf{r}_0 - \mathbf{r}) \cdot (\mathbf{v}_0 - \mathbf{v})) = 0 \end{cases} \tag{9.2}$$

are the possible motions of the particles.

Note that the properties (9.2) mean the kinematic equivalence of the motion of particles M_0 and M, and the motion of the ends of a rigid bar M_0M of length l.

The solution of the problem under investigation is eventually reduced to the construction of a closed system of differential equations

$$\begin{aligned} m_0 \ddot{\mathbf{r}}_0 &= \mathbf{F}_0 (\mathbf{r}_0 - \mathbf{r}, \mathbf{v}_0 - \mathbf{v}), \\ m \ddot{\mathbf{r}} &= \mathbf{F} (\mathbf{r}_0, \mathbf{v}_0, \mathbf{r}, \mathbf{v}). \end{aligned} \tag{9.3}$$

Here, the right-hand side $\mathbf{F}$ of the closure equation, which represents the required force, is determined from the following condition for the realization of the motion with the given properties (9.2):

$$\begin{aligned} &(\mathbf{v}_0 - \mathbf{v})^2 + ((\mathbf{r}_0 - \mathbf{r}) \cdot (\mathbf{F}_0 - \mathbf{F})) \\ &\quad = \Phi (\omega_1, \omega_2, \mathbf{r}_0, \mathbf{r}, \mathbf{v}_0, \mathbf{v}, t). \end{aligned} \tag{9.4}$$

It can be easily seen that the required force $\mathbf{F}$ is determined from this condition to within its two arbitrary projections onto the coordinate axes and one function $\Phi (\omega_1, \omega_2, \mathbf{r}_0, \mathbf{r}, \mathbf{v}_0, \mathbf{v}, t)$, which is also indeterminate

and satisfies only the condition $\Phi|_{\omega_1=\omega_2=0} = 0$. Therefore, the additional properties of motion of the particles, which do not contradict the properties (9.2), should be indicated in order to finally solve the problem formulated above.

Sec. 10. Construction of Canonical Equations of Motion

Let us consider the following problem [26]:

Given the properties of motion of a mechanical system, described by the compatible and independent equalities

$$\Omega: \omega_\mu (q, p, t) = c_\mu - \text{const} \quad (\mu = 1, \ldots, m \leqslant n), \tag{10.1}$$

where $q\,[q_1, \ldots, q_n]$ is the vector of generalized coordinates of the system, and $p\,[p_1, \ldots, p_n]$ is the vector of generalized momenta.

Construct the canonical equations

$$\begin{aligned} \dot{q} &= Q\,(q, p, t), \\ \dot{p} &= P\,(q, p, t) \end{aligned} \tag{10.2}$$

describing the motion of the mechanical system, the motion with the given properties (10.1) *being one of the possible motions of this system.*

Here, the given functions $\omega_\mu\,(q, p, t)$ are supposed to be bounded, continuous, and differentiable in some domain $G\,\{q, p\}$ for $t \geqslant t_0$; the right-hand sides of Eqs. (10.2) are sought in the class of functions admitting the existence of the solutions of these equations in some ε-neighbourhood Ω_ε of the manifold Ω defined by Eqs. (10.1). Further, the initial conditions $(q, p)_{t=t_0} \in \Omega$ cause the motion of the corresponding representative point $M\,(q, p)$ along the given manifold Ω.

The required equations are assumed to have the form

$$\begin{aligned} \dot{q} &= Q\,(q, p, t), \\ \dot{p} &= P^\tau\,(q, p, t) + P^\nu\,(q, p, t), \end{aligned} \tag{10.3}$$

where $P^\tau\,[P_1^\tau, \ldots, P_n^\tau]$ is the vector of generalized potenti-

al forces $P_i^\tau \doteq -\frac{\partial H}{\partial q_i}$ $(i = 1, \ldots, n)$, Q is the vector function with components $Q_i = \frac{\partial H}{\partial p_i}$ $(i = 1, \ldots, n)$, $H = H(q, p, t)$ is the Hamiltonian of the system, and $P^\nu [P_1^\nu, \ldots, P_n^\nu]$ is the vector of the active generalized nonpotential forces.

The necessary and sufficient conditions for the manifold (10.1) to be the integral manifold for the system of equations (10.3) are written in the form

$$(\omega_\mu, H) + (\operatorname*{grad}_p \omega_\mu \cdot P^\nu) = \Phi_\mu(\omega, p, q, t) - \frac{\partial \omega_\mu}{\partial t}, \qquad (10.4)$$

where (ω_μ, H) is Poisson's bracket composed for the functions ω_μ and H, and $\Phi_\mu(\omega, q, p, t)$ is an arbitrary function satisfying the condition $\Phi_\mu|_{\omega=0} = 0$ and $\Phi_\mu \equiv 0$ for $c_\mu \neq 0$.

Henceforth, we shall consider the motion of the representative point $M(q, p)$ along the integral manifold Ω (10.1) under the action of the generalized potential forces P^τ only. Therefore, the structure of these forces is defined by the equations (Poisson's theorem [11])

$$(\operatorname*{grad}_p \omega_\mu \cdot P^\tau) + (\operatorname*{grad}_q \omega_\mu \cdot Q) + \frac{\partial \omega_\mu}{\partial t} = 0 \quad (\mu = 1, \ldots, m), \qquad (10.5)$$

obtained from (10.4) for $P^\nu|_{\omega=0} = 0$, $\Phi_\mu|_{\omega=0} = 0$. Having solved Eqs. (10.5) for the components of the vector P^τ, we obtain

$$P_s^\tau = -\sum_{r=1}^{m} \frac{\Delta_{rs}}{\Delta}\left[(\operatorname*{grad}_q \omega_r \cdot Q) + \frac{\partial \omega_r}{\partial t}\right] - \sum_{k=m+1}^{n} \frac{\Delta^{sk}}{\Delta} P_k^\tau$$

$$(s = 1, \ldots, m), \qquad (10.6)$$

where $\Delta = \det \left\| \frac{\partial \omega_r}{\partial p_s} \right\|_m^m$, Δ_{rs} is the cofactor of the (r, s)-th element of the determinant Δ, and Δ^{sk} is a determinant obtained from Δ by replacing its s-th column $(s = 1, \ldots, m)$ by the k-th column $(k = m + 1, \ldots, n)$ of the matrix $\left\| \frac{\partial \omega_i}{\partial p_l} \right\|_n^m$.

Taking (10.5) into consideration, we get from (10.4)

$$(\operatorname{grad}_p \omega_\mu \cdot P^\nu) = \Phi_\mu \ (\omega, q, p, t) \quad (\mu = 1, \ldots, m). \qquad (10.7)$$

These equations allow us to determine the structure of the generalized nonpotential forces applied to a system, while the representative point is outside the given manifold (10.1). From these equations, we find that

$$P_s^\nu = \sum_{r=1}^{m} \frac{\Delta_{rs}}{\Delta} \Phi_r (\omega, q, p, t) - \sum_{k=m+1}^{n} \frac{\Delta^{sk}}{\Delta} P_k^\nu$$

$$(s = 1, \ldots, m). \qquad (10.8)$$

Thus, the required system of equations of motion, for which the given properties (10.1) describe one of its possible motions, is given by

$$\begin{aligned} \dot{q}_i &= Q_i (q, p, t) \quad (i = 1, \ldots, n), \\ \dot{p}_s &= \sum_{r=1}^{m} \frac{\Delta_{rs}}{\Delta} \left[\Phi_r - (\operatorname{grad}_q \omega_r \cdot Q) - \frac{\partial \omega_r}{\partial t} \right] \\ &\quad - \sum_{k=m+1}^{n} \frac{\Delta^{sk}}{\Delta_r} (P_k^\tau + P_k^\nu) \quad (s = 1, \ldots, m), \\ \dot{p}_k &= P_k^\tau + P_k^\nu \quad (k = m+1, \ldots, n). \end{aligned} \qquad (10.9)$$

It can be seen from (10.9) that these equations contain the indeterminate functions

$$Q_i (q, p, t) \quad (i = 1, \ldots, n),$$

$$\left. \begin{array}{l} P_k^\tau (q, p, t), \\ P_k^\nu (q, p, t) \end{array} \right\} \quad (k = m+1, \ldots, n),$$

$$\Phi_r (\omega, q, p, t) \quad (r = 1, \ldots, m; \ \Phi_r \equiv 0 \text{ for } c_s \neq 0).$$

Besides the above conditions, these functions must satisfy the condition that Eqs. (10.9) are canonical. This requirement can be written in the following form:

$$\frac{\partial Q_i}{\partial q_j} = -\frac{\partial P_j^\tau}{\partial p_i}, \quad \frac{\partial P_i^\tau}{\partial q_j} = \frac{\partial P_j^\tau}{\partial q_i}, \quad \frac{\partial Q_i}{\partial p_j} = \frac{\partial Q_j}{\partial p_i}$$

$$(i,\ j = 1, \ldots, n). \tag{10.10}$$

These are the conditions for the existence of Poincaré's integral invariant for the equations of motion of a mechanical system under the action of generalized potential forces [11].

It should be noted that in some inverse problems of dynamics, the right-hand sides $Q\ (q,\ p,\ t)$ of the first group of canonical equations are known a priori from the statement of the problem itself (for instance, the presence of kinematic equations in the inverse problems of the motion of a rigid body with one fixed point), and the solution of the problem is reduced to the determination of the generalized potential forces P_k^τ controlling the motion of the representative point along the integral manifold Ω (10.1). The remaining indeterminate functions P_k^ν and Φ_r can be found subsequently by imposing the additional requirements.

It should also be noted that the proposed technique of constructing the equations of motion imposes additional requirements on the given properties of motion in that $\Delta \neq 0$ for $\forall M\ (q,\ p) \in \Omega_\varepsilon$ for $t \geqslant t_0$. This requirement must be met while formulating the problem itself.

Let us consider some particular cases of this problem.

1. Suppose that $m = n$. In this case, the nonpotential forces are defined only by the functions $\Phi_r\ (\omega,\ q,\ p,\ t)$ $(r = 1, \ldots, n)$, and the required equations of motion are given by the following system:

$$\begin{aligned} \dot{q}_s &= Q_s\ (q,\ p,\ t) \quad (s = 1, \ldots, n), \\ \dot{p}_s &= \sum_{r=1}^{n} \frac{\Delta_{rs}}{\Delta} \left[\Phi_r - (\operatorname{grad}_q \omega_r \cdot Q) - \frac{\partial \omega_r}{\partial t} \right]. \end{aligned} \tag{10.11}$$

Here, if ω_μ does not depend on t, the equations of motion have the form

$$\begin{aligned} \dot{q}_s &= Q_s(q, p, t) \quad (s = 1, \ldots, n), \\ \dot{p}_s &= \sum_{r=1}^{n} \frac{\Delta_{rs}}{\Delta} [\Phi_r - (\operatorname*{grad}_q \omega_r \cdot Q)]. \end{aligned} \tag{10.12}$$

2. The properties of motion are given in the form $\omega_\mu \equiv \equiv p_\mu - f_\mu(q, t) = 0$. The possibility of defining the properties of motion in such a form directly follows from Jacobi's theorem concerning the first integrals of the canonical equations.

Let us first suppose that $m < n$. Then the required equations of motion form the system

$$\begin{aligned} \dot{q}_i &= Q_i(q, p, t) \quad (i = 1, \ldots, n), \\ \dot{p}_s &= (\operatorname*{grad}_q f_s \cdot Q) + \frac{\partial f_s}{\partial t} + \Phi_s \; (s = 1, \ldots, m), \\ \dot{p}_k &= P_k^\tau + P_k^\nu \quad (k = m+1, \ldots, n). \end{aligned} \tag{10.13}$$

Assuming further that $m = n$, the equations of motion acquire the form

$$\begin{aligned} \dot{q}_s &= Q_s(q, p, t) \quad (s = 1, \ldots, n), \\ \dot{p}_s &= (\operatorname*{grad}_q f_s \cdot Q) + \frac{\partial f_s}{\partial t} + \Phi_s. \end{aligned} \tag{10.14}$$

3. The properties of motion are prescribed in the form $\omega_\mu(q_\mu, p_\mu) = c_\mu$. The possibility of defining the equations of motion in such a form arises from the fact that in the case of separation of variables in Jacobi's equation, some of the integrals in the canonical equations happen to be in this form.

If $m < n$, the required equations of motion form the system

$$\begin{aligned} \dot{q}_i &= Q_i(q, p, t) \quad (i = 1, \ldots, n), \\ \dot{p}_s &= -\left(\frac{\partial \omega_s}{\partial q_s} \Big/ \frac{\partial \omega_s}{\partial p_s}\right) Q_s + \Phi_s \quad (s = 1, \ldots, m), \\ \dot{p}_k &= P_k^\tau + P_k^\nu \quad (k = m+1, \ldots, n). \end{aligned} \tag{10.15}$$

If $m = n$, the equations of motion have the form

$$\dot{q}_s = Q_s(q, p, t) \quad (s = 1, \ldots, n),$$
$$\dot{p}_s = -\left(\frac{\partial \omega_s}{\partial q_s} \Big/ \frac{\partial \omega_s}{\partial p_s}\right) Q_s + \Phi_s. \tag{10.16}$$

It should be noted that if all $Q_1, \ldots, Q_n$ are known, and $m = n$, then a part of the conditions (10.10) is imposed only on the integral manifold Ω (10.1). This restricts the possibilities of the proposed technique of constructing the equations of motion as compared to the general methods suggested above. A natural explanation of such a limitation is that this technique is proposed only for constructing a class of equations of motion defined beforehand, namely, the canonical equations.

Sec. 11. A Rigid Body with One Fixed Point

By way of an example, let us examine the construction of canonical equations of motion of a symmetrical ($A = B$) rigid body with one fixed point, assuming that the properties of motion are given by

$$\Omega: \begin{cases} \omega_1 \equiv \frac{1}{2}\left[\frac{(p_1 - p_3\cos\theta)^2}{A\sin^2\theta} + \frac{p_2^2}{A} + \frac{p_3^2}{C}\right] - U = c_1 & (11.1) \\ \omega_2 \equiv p_1 = c_2 & (11.2) \\ \omega_3 \equiv p_3 - Cr_0 = 0, & (11.3) \end{cases}$$

where p_1, p_2, p_3 are the generalized momenta corresponding to Euler's angles ψ, θ, φ; A, B, C are the moments of inertia of the rigid body about the principal axes x, y, z with the origin at the fixed point; $U = U(\psi, \theta, \varphi)$ is the force function, and r_0 is the given velocity of rotation of the body around the z-axis.

In this case, the first group of canonical equations is given by the system

$$\begin{aligned} \dot{\psi} &= \frac{p_1 - p_3\cos\theta}{A\sin^2\theta}; \\ \dot{\theta} &= \frac{p_2}{A}, \\ \dot{\varphi} &= \frac{p_3}{C} - \frac{p_1 - p_3\cos\theta}{A\sin^2\theta}\cos\theta, \end{aligned} \tag{11.4}$$

which directly follows from the well-known structure of the Hamiltonian defined by the energy integral $\omega_1 = c_1$.

The remaining unknown equations of motion are constructed in accordance with (10.12), and can be represented in the form

$$\dot{p}_1 = 0,$$
$$\dot{p}_2 = \frac{A}{p_2}\left(\frac{\partial U}{\partial \psi}\dot{\psi} + \frac{\partial U}{\partial \varphi}\dot{\varphi}\right) - \left[\frac{(p_1 - p_3\cos\theta)(p_3 - p_1\cos\theta)}{A\sin^3\theta} - \frac{\partial U}{\partial \theta}\right] - \frac{A}{p_2}\dot{\varphi}\Phi, \tag{11.5}$$
$$\dot{p}_3 = \Phi,$$

where Φ is an arbitrary function of the generalized coordinates and momenta, vanishing at $p_3 = Cr_0$.

The conditions (10.10) allow us to form an opinion about the possible structure of the force function U, for which the equations of motion (11.4) and (11.5) admit the first integrals $\omega_1 = c_1$ and $\omega_2 = c_2$, and the rigid body can have a given constant velocity of rotation.

As a matter of fact, for the problem under investigation, it follows directly from (10.10) that

$$\frac{\partial U}{\partial \psi}\dot{\psi} + \frac{\partial U}{\partial \varphi}\dot{\varphi} = 0, \tag{11.6}$$

and hence we can take an arbitrary function, depending only on the angle of nutation θ, as the force function.

Thus, the equations of motion (11.5) are finally written in the form

$$\dot{p}_1 = 0,$$
$$\dot{p}_2 = -\left[\frac{(p_1 - p_3\cos\theta)(p_3 - p_1\cos\theta)}{A\sin^3\theta} - \frac{\partial U}{\partial \theta}\right] - \frac{A}{p_2}\left(\frac{p_3}{C} - \frac{p_1 - p_3\cos\theta}{A\sin^2\theta}\cos\theta\right)\Phi, \tag{11.7}$$
$$\dot{p}_3 = \Phi.$$

The function Φ appearing in these equations defines the nonpotential generalized forces applied to a body, when the velocity $\dot{\varphi}$ of free rotation of the rigid body is different from its given value r_0.

Chapter Two

CONSTRUCTION OF STABLE SYSTEMS

One of the classical inverse problems of dynamics is the analytical construction of stable mechanical systems, i.e. the determination of the parameters of a system and the applied generalized forces. In this case, one of the (indicated a priori) possible motions of the system (unperturbed motion) should be stable with respect to some (also indicated a priori) characteristic indices of motion in the presence of initial, constantly acting, and parametric perturbations.

A mathematical formulation of the idea of stability, possible statements of the problems of stability of equilibrium and the motion of mechanical systems, as well as the principal methods of solving these problems had already been established in the last century in the famous works of Lagrange, Routh, Zhukovskii, Lyapunov, and Poincaré. Naturally, besides the problems of investigating the stability of motion of the given mechanical systems, the methods of constructing stable systems were also described in these works. Thus, for example, the well-known sufficient conditions of stability, obtained by Lyapunov [27], are essentially the initial criteria for selecting the parameters of the system, or for determining the additional forces under which a given motion is stable in the presence of initial perturbations.

In the subsequent fundamental works on the theory of stability, the problem of construction of stable systems clearly emerged as one of the basic problems of this theory. The efficiency of the methods of the theory of stability was shown while solving the most difficult applied problems regarding the construction of stable material systems of mechanical or other type.

At present, the theory of construction of stable systems is one of those branches of science on controlled motions of material systems, which are richly supplemented by new special investigations by a large number of mathematicians, scientists, and engineers. The procedure of solving the problems regarding the construction of stable systems is being continuously enriched, and the field of application of the solutions of these problems is constantly growing.

In this chapter, the problem of the analytical construction of stable material systems is regarded, at least in the initial stage of its solution, as one of the inverse problems, namely, the problem of construction of equations of motion in such a way that the given motion of a system should possess the property of stability of some (also specified) indices of motion in the presence of initial perturbations. We shall describe the methods of analytical construction of stable material systems on the basis of the methods of characteristic numbers and Lyapunov's functions, which have wide possibilities in the problems on constructing stable systems. To begin with, we consider the basic ideas and definitions in the theory of stability of material systems, and a general review of the basic sufficient conditions of stability of motion necessary for the solution of inverse problems of dynamics.

We shall also formulate in this chapter the problem of determination of a set of functions relative to which a given motion, forming the basis for constructing the appropriate equations of motion of a material system, is a stable motion of the system under investigation. Besides, we shall also discuss the solution of the formulated problem with the help of the methods of characteristic numbers. It should be noted that the solution of this problem with the subsequent interpretation of the reference functions obtained here allows us to determine a set of geometrical and kinematic indices of motion of a material system relative to which the given motion turns out to be stable.

Sec. 1. Basic Concepts of the Theory of Stability

Let us consider a material system whose possible motions are described by the equations

$$\mathscr{L}(q, \dot{q}, \ddot{q}, t) = 0, \tag{1.1}$$

where $q\ [q_1, \ldots, q_N]$ is the vector of generalized coordinates of the system, $\mathcal{L}\ [\mathcal{L}_1, \ldots, \mathcal{L}_N]$ is the vector function for which the given domain $G\ \{q, \dot{q}, \ddot{q}\} \times T\ \{t \geqslant t_0\}$ is the domain of existence of solutions of the equations of motion (1.1).

Let us suppose that we know one of the particular solutions

$$q = \varphi\,(t;\ q_0,\ \dot{q}_0,\ t_0) \tag{1.2}$$

of Eqs. (1.1), satisfying the initial conditions

$$\begin{aligned} \varphi\,(t;\ q_0,\ \dot{q}_0,\ t_0)\,|_{t=t_0} &= q_0, \\ \dot{\varphi}\,(t;\ q_0,\ \dot{q}_0,\ t_0)\,|_{t=t_0} &= \dot{q}_0. \end{aligned} \tag{1.3}$$

This solution describes one of the possible motions of the system, corresponding to the initial given values of the generalized coordinates q_0 and generalized velocities $\dot{q}_0$.

Suppose that we are interested somehow in the motion of a system described by this solution (this may be the desirable motion, for example). We call this motion the *unperturbed motion*. All the other possible motions of the system are called *perturbed motions*.

It should be noted that the unperturbed motion of a system, as well as its perturbed motions, are the possible motions of the system under the action of the same generalized forces and for the same parameters of the system. The unperturbed motion is caused by the completely defined possible initial state of the system (1.3), while all the perturbed motions are caused by other possible initial states of the system like the initial values of the generalized coordinates and velocities, which differ from (1.3) and are in fact random quantities.

The initial deviations of the generalized coordinates and velocities of a system in perturbed motions from their corresponding values (1.3) in unperturbed motion are called the *initial perturbations of the coordinates and velocities.*

The accepted concept of the stability of motion includes first of all the inflexibility of any characteristic indices of motion of a system under possible initial perturbations of the coordinates and velocities. Here, it is necessary that these characteristic indices for all the perturbed motions

be close to their corresponding values for the unperturbed motion. In this way, the strength of the unperturbed motion of the system is established in a certain sense.

It should be noted that characteristic indices of motion of a system as well as its unperturbed motion itself are fixed a priori in the formulation of the problem of stability. Geometrical and kinematic elements of motion, and in particular, the generalized coordinates and velocities, can be used as these indices. In the general case, the characteristic indices of motion are expressed as some functions of the generalized coordinates and velocities, as well as time,

$$Q_s(q, \dot{q}, t) \quad (s = 1, \ldots, n), \tag{1.4}$$

defined in the domain of existence of the solutions of the system of differential equations (1.1) and differentiable with respect to all the variables in the same domain. Let us call these functions the *reference functions.*

An investigation of the stability of the unperturbed motion (1.2) of a system with respect to the characteristic indices (1.4) means the investigation of deviations of the values of these indices for the perturbed motions from their corresponding values for the unperturbed motion as a function of time, beginning with some moment t_0. In order to do so, we form the differences between the values of reference functions (1.4) for the perturbed motions defined by (1.1), and for the chosen a priori unperturbed motion (1.2) defined by the initial conditions (1.3):

$$x_s = Q_s(q, \dot{q}, t) - Q_s(\varphi, \dot{\varphi}, t) \quad (s = 1, \ldots, n). \tag{1.5}$$

We call these differences the *perturbations*, and the vector $x\,[x_1, \ldots, x_n]$ the *perturbation vector.*

Mathematically, the problem of investigating the stability is reduced to the investigation of the change in the perturbation vector x as a function of time according to the differential equations (1.1).

Naturally, a direct determination of the perturbations $x_1, \ldots, x_n$ is possible only in quite special cases, when the system of equations (1.1) allows us to find the general solution.

In order to investigate the stability in the general case, we first form the expressions for the derivatives of perturbations

$$\dot{x}_s = \frac{\partial Q_s(q, \dot{q}, t)}{\partial t} + \sum_{\nu=1}^{n} \left(\frac{\partial Q_s(q, \dot{q}, t)}{\partial q_\nu} \dot{q}_\nu + \frac{\partial Q_s(q, \dot{q}, t)}{\partial \dot{q}_\nu} \ddot{q}_\nu \right)$$
$$- \dot{Q}_s(\varphi, \dot{\varphi}, t) \quad (s = 1, \ldots, n). \quad (1.6)$$

Next, the variables q, $\dot{q}$, $\ddot{q}$ are eliminated from these equations with the help of the equations of motion (1.1) and the definition of the perturbations themselves (1.5). Thus, the system of differential equations in perturbations is constructed:

$$\dot{x} = X(x, t). \quad (1.7)$$

It is called the system of *equations of the perturbed motion*. This system has a trivial solution $x = 0$, corresponding to the unperturbed motion (1.2).

Thus, an investigation of the stability of motion of material systems is reduced to the solution of one of the mathematical problems, namely, to the investigation of the behaviour of solutions of systems of differential equations (1.7) for $t \geqslant t_0$. Therefore, it is necessary to give a mathematical definition to the very idea of stability.

The definition of the stability of motion, given by Lyapunov [27] as the most general definition which predetermines not only the scope and content of the problems embraced by the modern theory of stability, but also the development of the qualitative methods of investigating differential equations for solving these problems, forms the basis of the modern theory of stability.

The definition of stability according to Lyapunov [28] is given as follows:

If for any arbitrarily given positive number ε, *however small, there exists a number* δ, *such that the inequality*

$$\sum_{s=1}^{n} x_s^2 < \varepsilon \quad (1.8)$$

holds for any initial perturbations (for $t = t_0$*), satisfying the condition*

$$\sum_{s=1}^{n} x_{s0}^2 \leqslant \delta, \tag{1.9}$$

and for any $t \geqslant t_0$*, the unperturbed motion* (1.2) *is stable with respect to the characteristic indices* (1.4). *Otherwise, the unperturbed motion* (1.2) *is unstable. If an unperturbed motion is stable and*

$$\lim_{t\to\infty} \sum_{s=1}^{n} x_s^2 = 0, \tag{1.10}$$

the stability is asymptotic.

In the general case, the stability or instability of motion of the same material system under the action of the same generalized forces and for the same values of the parameters depends on the choice of the unperturbed motion (1.2), the reference functions (1.4), and the initial moment of time t_0. This result follows from the above definition of stability.

We can give examples, when one of the possible motions of the material system is stable relative to some given characteristic indices, while the other motions are unstable relative to the same indices (for instance, stable and unstable operation of the same engine).

Conversely, the same motion of a material system can be stable relative to some characteristic indices and unstable relative to others (for instance, instability of motion of planets relative to Cartesian or polar coordinates, and its stability relative to the area integral).

The dependence of stability on the choice of the initial moment implies that it is necessary to introduce in the theory of stability the idea of uniform stability [29], when we impose the additional requirement concerning the independence of the stability of the choice of the initial moment.

Depending on the ultimate aim, the problems of stability can be formulated in different forms. Thus, for example, we can pose the problem of establishing the stability or instability only of some completely defined system (analysis of the stability of a system) [27-30]. Similarly, we can also

pose the problem of construction of material systems which have stable motions, and have preset properties (programmed motion) [13, 31].

However, irrespective of the stability problem, its analytical solution requires that we must establish beforehand

(1) the equations of motion (1.1) of the material system under investigation (mathematical model of the system);

(2) the unperturbed motion (1.2), whose stability is investigated (particular solution or the integral manifold of the equations of motion);

(3) the reference functions (1.4), representing the characteristic indices of motion, relative to which the stability is investigated; and

(4) the initial moment of time (usually $t_0 = 0$).

In accordance with the very definition of stability, we can pose any problem of stability only after the above requirements have been met.

It was suggested by Lyapunov that there can be two basic methods of investigating the stability of motion of material systems [27, 28]. The first of these methods is based on the construction of the solution of a nonlinear system of equations (1.7) of unperturbed motion in the form of a special type of series, using the solution of the corresponding equations of the perturbed motion in the first approximation. Therefore, the behaviour of the solution of nonlinear equations (1.7) of the perturbed motion is determined by the behaviour of the solution of the corresponding linear system, in the general case, with coefficients varying in time. But the behaviour of the solutions of a linear system is investigated in comparison with the behaviour of the function $\exp \lambda_0 t$, where λ_0 is a constant, called the characteristic number of the solution of the corresponding linear system. Here, the right-hand sides of the equations (1.7) of perturbed motion are supposed to be expanded into convergent series in the powers of the perturbations $x_1, \ldots, x_n$ themselves in some domain

$$H \left\{ \sum_{s=1}^{n} x_s^2 \leqslant H \right\}, \tag{1.11}$$

the coefficients of expansion being bounded continuous functions of time $t \geqslant t_0$. Hence, the equations of the per-

turbed motion can be represented in the form

$$\dot{x}_s = \sum_{k=1}^{n} p_{sk}(t)\, x_k + \sum_{m_1+\ldots+m_n \geqslant 2} P_s^{(m_1, \ldots, m_n)}(t)\, x_1^{m_1} \ldots x_n^{m_n} \quad (s = 1, \ldots, n), \tag{1.12}$$

where $m_1, \ldots, m_n$ are natural numbers.

In future, the idea of characteristic numbers will be used in order to solve the problem of stability in accordance with the equations of the first approximation,

$$\dot{x}_s = \sum_{k=1}^{n} p_{sk}(t)\, x_k \quad (s = 1, \ldots, n), \tag{1.13}$$

taking into account the terms of the highest orders in the entire system of equations of the perturbed motion (1.12).

Nowadays, this method of investigating the stability is called the *method of Lyapunov's indices* (method of characteristic numbers).

The idea behind the second method suggested by Lyapunov is that the conclusions about the stability or instability are made according to the properties of some specially constructed function $V(x, t)$, called Lyapunov's function, and its derivative formed with the help of the system of equations (1.7) of the perturbed motion

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s(x, t). \tag{1.14}$$

This method of investigating the stability is called the *method of Lyapunov's functions.*

In conclusion of this section, let us remark that the very definition of stability of a motion as well as the methods of its investigation were proposed by Lyapunov for the case of initial perturbations only. Naturally there also exist at present some other views on the idea of stability (stability in the presence of constantly acting and parametric perturbations, stability in general, absolute stability, stability during a finite interval of time, and so on). Different views on the idea of stability also exist in the works of Lyapunov's predecessors (stability in Poisson's sense, Zhukovskii's orbital stability, and so on).

All these interpretations of stability are completely covered in Lyapunov's treatment, namely, the inflexibility of a definite motion of material systems under various types of perturbations.

As regards the methods proposed by Lyapunov, for the investigation of stability, they have been widely developed while solving the simple problems of stability, as well as while constructing various material systems of different physical nature and structure, which are stable in some sense. Lyapunov's methods continue to be the basic methods in the modern theory of stability.

Sec. 2. Formulation of the Problem of Constructing Stable Systems

Consider a material system whose motions (the changes in its state) are described by ordinary differential equations

$$\mathscr{L}(q, \dot{q}, \ddot{q}, t, u) = 0, \tag{2.1}$$

where $q\,[q_1, \ldots, q_N]$ is the vector of state of the system (q_i are the generalized coordinates of the system, $i = 1, \ldots$ $\ldots, N$); $u\,[u_1, \ldots, u_m]$ is the system control vector (u_μ can be the eigenparameters of the system or the generalized forces controlling the motion of the system, $\mu = 1, \ldots, m$); $\mathscr{L}\,[\mathscr{L}_1, \ldots, \mathscr{L}_N]$ is the vector function defined in a given domain $G\,\{q, \dot{q}, \ddot{q}\} \times T\,\{t \geqslant t_0\} \times U\,\{u\}$ which is the domain of existence of the solutions of the system of equations (2.1).

We pose the following problem:

Determine the system parameters and the controlling forces in the form of functions of time $u_1(t), \ldots, u_m(t)$ in such a way that the given motion of the system

$$\Omega\colon\ q = \varphi(t), \quad \varphi(t) \in C^2\,\{t \geqslant t_0\} \tag{2.2}$$

is one of the possible motions of the material system under investigation, and is stable relative to the given characteristic indices of motion (reference functions)

$$Q\,[Q_1, \ldots, Q_n], \quad Q_\nu(q, \dot{q}, t) \in C^1 \qquad (\nu = 1, \ldots, n) \tag{2.3}$$

in the presence of arbitrarily small deviations of the values of these indices from their values on the given motion (2.2) *at the initial moment of time* t_0.

Here, it is assumed that the functions $u_1(t)$, . . ., $u_2(t)$ are sought in the class of bounded and continuous functions also satisfying the conditions of uniqueness of the solution of the system of equations (2.1) at least in the ε-neighbourhood Ω_ε of the given curve Ω (2.2) in the space of generalized coordinates.

We can see that this problem is an inverse problem of dynamics, which we earlier designated as the restoration problem. However, the equations of motion must be restored here not only from the requirement that the given motion be one of the possible motions of the system, but also by taking into account the additional requirement that the given motion be stable. Moreover, the stability must be ensured only by a selection of the vector components—controlling force u (t)—without changing the structure of the equations of motion (2.1).

It should be noted that even if the given motion of the material system is one of its possible motions, it is still not possible to realize it in practice, since the coincidence of the initial conditions of the real motion of a system with the initial conditions of the given motion of the system is still required. In fact, the motions of the system can occur, and as a rule, they do occur under the initial conditions which are different from the given conditions. That is why the requirement of the stability of the given motion in the presence of the initial perturbations is imposed, in addition to the requirement of its realization, in the inverse problem of dynamics.

Thus, the problem must be solved in two stages.

First of all, the inverse problem of dynamics is solved under the assumption that the given motion (2.2) of the system is realized exactly. For this purpose, the following necessary conditions are imposed for the realization of the given motion:

$$\mathscr{L}\ (\varphi\ (t),\ \dot{\varphi}\ (t),\ \ddot{\varphi}\ (t),\ t,\ u) = 0. \tag{2.4}$$

The very formulation of the problem makes these conditions compatible. Suppose that these conditions form an indefinite system of equations in the components of the control vector

$u(t)$. Then, these conditions will determine with some reservations the required parameters and the controlling forces for which the given motion (2.2) is one of the possible motions of the material system under investigation.

After this, we must solve the problem of ensuring the stability of the given motion (2.2) relative to the given characteristic indices of the motion (2.3) in the presence of initial perturbations. For this purpose, we use the remaining freedom of the choice of the components of the control vector $u(t)$. First of all, the appropriate equations of the perturbed motion are constituted under the assumption that the motion (2.2) is the unperturbed motion of the system. In the case under consideration, these equations have the following form:

$$\dot{x} = X(x, t, u), \tag{2.5}$$

where $x[x_1, \ldots, x_n]$ is the perturbation vector ($x_\nu = Q_\nu(q, \dot{q}, t) - Q_\nu(\varphi(t), \dot{\varphi}(t), t)$, $\nu = 1, \ldots, n$); $X(x, t, u)$ is a vector function vanishing at $x = 0$, corresponding to the given motion (2.2).

Having done this, we establish the sufficient conditions for the stability of the trivial solution $x = 0$ of the system (2.5) of the equations of the perturbed motion, with the help of some method of investigating the stability. In the general case, these conditions will be represented in the form of some inequalities in the components of the control vector $u(t)$:

$$S(\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), t, u) \geqslant 0. \tag{2.6}$$

Finally, the conditions of stability (2.6), together with the conditions of realization (2.4), allow us to determine the unknown parameters of the system and the controlling forces in such a way that the given motion (2.2) turns out to be one of the possible motions of the material system (2.1) under investigation, and stable with respect to the given characteristic indices of the motion (2.3) in the presence of initial perturbations.

In conclusion, it should be noted that the problem formulated above is not the only possible or the most general problem of constructing stable systems. It should be considered only as one of the classical problems of restoration

of equations of motion, taking into account the additional requirement of the stability of the given motion. Here, this problem must be the initial problem in the analytical construction of an open-loop control system in accordance with the given stable law of motion of this system [14, 22].

Naturally, we can also formulate the problem of closure of the equations of motion of some object, having mechanical or some other physical nature, with the additional requirement of the stability of the given motion of this object. This problem will be the initial problem in the analytical construction of regulators [14, 22].

The statement of the problem of constructing the equations of motion of a material system in accordance with the given stable properties of its motion under the assumption that the control vector is defined as a function of the system coordinates and time will be a more general statement of this type of problem. Such a formulation is examined in the theory of construction of the system of programmed motion [15, 22].

Finally, the methods of finding the conditions of stability of motion have to be used for the solution of the formulated problem as well as its various modifications and generalizations. Therefore, we shall describe the methods formulated by Lyapunov and his successors for investigating the stability, which are also applicable for solving the formulated problem on the construction of stable systems.

Sec. 3. Application of the Method of Characteristic Numbers

Let us suppose that in the process of the solution of the above problem on the construction of stable systems, the appropriate equations of the perturbed motion are given in the form

$$\dot{x}_s = \sum_{k=1}^{n} p_{sk}(t, u)\, x_k + X_s^{(2)}(x, t, u) \quad (s = 1, \ldots, n), \tag{3.1}$$

where $p_{sk}(t, u)$ are bounded functions, continuous in the domain $T\,\{t \geqslant t_0\} \times U\,\{u\}$, $X_s^{(2)}(x, t, u)$ are holomorphic functions in the domain $H\,\{\sum_{s=1}^{n} x_s^2 \leqslant H\} \times T\,\{t \geqslant t_0\} \times$

$\times\ U\ \{u\}$, whose expansion in the powers of $x_1, \ldots, x_n$ begins with a term of not lower than the second order.

It should be noted that a material system is called a nonlinear, *nonstationary system* if the equations of the perturbed motion of this system are nonlinear and explicitly contain the time t (as in the present case). The unperturbed motion itself, $x = 0$ (the given motion in the present case), is called the *unsteady motion* of this system.

A system is called *stationary* if the equations of perturbed motion do not explicitly contain the time t. The unperturbed motion itself is called *steady-state motion.*

In the classical theory of stability, the problems of investigation of the stability of unsteady motion of nonlinear systems are stated and solved with the help of the first-approximation equations of the perturbed motion. While constructing stable systems, the appropriate problem can be formulated as follows:

Given the first-approximation equations of perturbed motion

$$\dot{x}_s = p_{s1}(t, u)\, x_1 + \ldots + p_{sn}(t, u)\, x_n \quad (s = 1, \ldots, n), \tag{3.2}$$

find the control vector $u\ (t)$ so that the given motion, defined as the unperturbed motion $x = 0$, is stable if we also consider terms of the highest order $X_s^{(2)}(x, t, u)$ $(s = 1, \ldots, n)$ in the equations of the perturbed motion (3.1).

In order to solve this problem, let us first examine the linear system of differential equations

$$\dot{x}_s = p_{s1}(t)\, x_1 + \ldots + p_{sn}(t)\, x_n \quad (s = 1, \ldots, n), \tag{3.3}$$

where $p_{sk}(t)$ are bounded functions, continuous for $t \geqslant t_0$. We introduce the required definitions and concepts of the theory of stability in connection with the formulated problem.

3.1. Characteristic Numbers of the Solutions of Equations. Suppose that we know a certain fundamental matrix of the solutions of the system of equations (3.3):

$$X = \| x_{sk}(t) \|_n^n, \tag{3.1.1}$$

where $x_{sk}(t)$ is the k-th solution of the system of equations $(s = 1, \ldots, n)$ in question. Let us single out one of the

solutions of the system of equations (3.3), for example, the k-th solution:

$$x_{1k}(t), \ldots, x_{nk}(t).$$

Then the number

$$\chi_k = \min\left\{-\overline{\lim_{t\to\infty}} \frac{\ln|x_{1k}(t)|}{t}, \ldots, -\overline{\lim_{t\to\infty}} \frac{\ln|x_{nk}(t)|}{t}\right\} \tag{3.1.2}$$

is called the *characteristic number of the k-th solution* of the system of equations (3.3). Thus, n characteristic numbers $\chi_1, \ldots, \chi_n$ are in accordance with the fundamental solution (3.1.1) of the system of linear equations (3.3).

The characteristic numbers are called *strict* [38] if the upper limits, indicated in (3.1.2) are finite and coincide with the corresponding lower limits. In this case, the characteristic number of the k-th solution of the system of equations (3.3) is defined as the lowest of the limits

$$-\lim_{t\to\infty} \frac{\ln|x_{sk}(t)|}{t} \quad (s = 1, \ldots, n).$$

Note that the set of the characteristic numbers of solutions of the system of equations (3.3) is defined eventually by selecting the fundamental system (3.1.1) of the solutions. However, the characteristic numbers of the solutions of the system of equations (3.3) under investigation have properties which are independent of the choice of any fundamental system of solutions, but are dictated only by the linear nature of these equations and the boundedness and continuity of their coefficients for $t \geqslant t_0$.

We shall enumerate these properties [27, 28].

1. Any solution of the system of equations (3.3), differing from the trivial solution $x = 0$, has a finite characteristic number.

2. The system of equations (3.3) cannot have more than n solutions with different characteristic numbers.

3. The sum of the characteristic numbers of the solutions of the system of equations (3.3) satisfies the inequality

$$\sum_{k=1}^{n} \chi_k \leqslant \chi\left\{\exp \int_{t_0}^{t} \sum_{s=1}^{n} p_{ss}(t)\, dt\right\}, \tag{3.1.3}$$

where $\chi\{f(t)\} = -\overline{\lim_{t\to\infty}} \frac{\ln|f(t)|}{t}$ is the characteristic number of the function $f(t)$.

Note that neither the definition of the characteristic numbers of the solutions of a system of linear differential equations, nor their properties mentioned above indicate a method of finding the characteristic numbers. A direct calculation of characteristic numbers of the solutions of the system of equations (3.3) is possible only in exceptional cases, when the corresponding fundamental system of solutions (3.1.1) has been found. However, while solving the stability problem with the help of the method of characteristic numbers, it is necessary either to establish the bounds for these numbers, or just to determine their signs. In particular, while solving the problems of construction of stable systems, the positiveness of characteristic numbers is the only requirement.

Let us consider some theorems which allow us to obtain such estimates for characteristic numbers.

Theorem 3.1.1 [28]. *The characteristic numbers of the solutions of a system are positive if the coefficients $p_{sk}(t)$ of the system of equations* (3.3) *are such that the principal diagonal minors of the determinant*

$$\det \| p_{sk}(t) + p_{ks}(t) \|_n^n \tag{3.1.4}$$

are sign-alternating, the function $p_{11}(t)$ being negative for $t \geqslant t_0$.

In particular, the negativeness of the diagonal coefficients $p_{ss}(t)$ $(s = 1, \ldots, n)$ for $t \geqslant t_0$ is the sufficient condition for the positiveness of characteristic numbers of the solutions of a system of equations with a skew-symmetric matrix of coefficients.

Theorem 3.1.2 [28]. *The characteristic numbers for the solutions of the system of equations* (3.3) *satisfy the inequalities*

$$\chi\left\{\exp\int_{t_0}^{t} \alpha(t)\,dt\right\} \geqslant \chi\{x\} \geqslant \chi\left\{\exp\int_{t_0}^{t} \beta(t)\,dt\right\}, \tag{3.1.5}$$

where $\alpha(t)$ and $\beta(t)$ are the minimum and maximum roots of the equation

$$\det \left\| \frac{p_{sk}(t) + p_{ks}(t)}{2} - \delta_{sk}\lambda \right\|_n^n = 0 \tag{3.1.6}$$

for all $t \geqslant t_0$.

Hence the inequality

$$\chi\left\{\exp\int_{t_0}^{t}\beta(t)\,dt\right\}>0$$

is sufficient for the positiveness of the characteristic numbers of the solutions of the system of equations (3.3).

More precise boundaries can be obtained for the appropriate characteristic numbers and even their values can be calculated in terms of the coefficients of the system of equations (3.3) in particular cases, when these coefficients satisfy some additional conditions. The estimates of the characteristic numbers of this type are indicated in the following theorems [32].

Theorem 3.1.3. *If for sufficiently large values of $t \geqslant t_0$, the coefficients of the system of equations* (3.3) *satisfy the inequality*

$$p_{11}(t) \geqslant P(t) + 2(n-1)Q(t), \tag{3.1.7}$$

where $P(t) = \max\{p_{ss}(t)\}$ $(s = 2, \ldots, n)$, $Q(t) = \max\{p_{sk}(t)\}$ $(s, k = 1, \ldots, n;\ s \neq k)$, *then the minimum characteristic number of the solutions of this system lies within the limits*

$$-\overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t}[p_{11}(t)+(n-1)Q(t)]\,dt \tag{3.1.8}$$

and

$$-\overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t}[p_{11}(t)-(n-1)Q(t)]\,dt. \tag{3.1.9}$$

Theorem 3.1.4. *The minimum characteristic number of the solutions of the system of equations* (3.3) *is equal to the limit*

$$-\overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t}p_{11}(t)\,dt \tag{3.1.10}$$

if for $t \geqslant t_0$ the coefficients of this system satisfy the conditions

$$\lim_{t\to\infty}p_{sk}(t)=0 \quad (s, k=1, \ldots, n;\ s\neq k), \tag{3.1.11}$$

$$p_{11}(t) \geqslant p_{ss}(t) + \varepsilon \quad (s = 2, \ldots, n), \tag{3.1.12}$$

where ε is a positive constant.

Theorem 3.1.5. *If for $t \geqslant t_0$ the coefficients of the system of equations* (3.3) *satisfy the conditions*

$$\lim_{t\to\infty} p_{sk}(t) = 0 \quad (s, k = 1, \ldots, n; \ s \neq k), \tag{3.1.13}$$

$$p_{s-1,s-1}(t) \geqslant p_{ss}(t) + \varepsilon \quad (s = 2, \ldots, n), \tag{3.1.14}$$

where ε is a positive constant, the characteristic numbers of the solutions of this system are equal to the limits

$$-\overline{\lim_{t\to\infty}} \frac{1}{t} \int_{t_0}^{t} p_{ss}(t)\,dt \quad (s = 1, \ldots, n). \tag{3.1.15}$$

It should be noted that the systems of linear differential equations, where the nondiagonal elements of the matrix of the coefficients vanish as $t \to \infty$, are called nearly diagonal. We can see that a direct calculation of the value of the minimum characteristic number of the solutions of this system is possible in the case of nearly diagonal systems if the condition (3.1.14) is satisfied. This allows us to obtain less stringent conditions for the positiveness of the characteristic numbers of these systems.

Sometimes, the characteristic numbers of the solutions of some system of equations are estimated in terms of the characteristic numbers of the solutions of another system, defined somehow by the given system. The following theorem [28, 33] illustrates one of these possibilities.

Theorem 3.1.6. *The characteristic numbers of the solutions of the system of equations* (3.3) *coincide with the characteristic numbers of the solutions of the limiting system of equations, if the coefficients $p_{sk}(t)$ in this system tend to definite limits c_{sk} as $t \to \infty$.*

This theorem permits us to reduce the problem of determining the characteristic numbers of the solutions of a system of equations with variable coefficients to a simpler problem of determining the characteristic numbers of the solutions of the system of equations with constant coefficients, the solution of which is established by the following theorem [27].

Theorem 3.1.7. *The characteristic numbers of the solutions of a system of equations with constant coefficients,*

$$\dot{x}_s = a_{s1}x_1 + \ldots + a_{sn}x_n, \tag{3.1.16}$$

are equal to the real parts of the roots of the corresponding characteristic equation

$$D(p) \equiv (-1)^n \det \| a_{sk} - \delta_{sk} p \|_n^n = 0, \qquad (3.1.17)$$

taken with the opposite sign.

3.2. The Regular System of Equations. It has been established above that the characteristic numbers of solutions of the system (3.3) of linear differential equations depend on the choice of the fundamental matrix (3.1.1) of the solutions of this system.

By constructing different fundamental matrices with the help of linear combinations of the available particular solutions, it is possible that the sum of all the n characteristic numbers of the fundamental matrix of the solution obtained in this way attains its maximum value σ, which exists in view of the boundedness of the characteristic numbers themselves [27]. Such a situation arises, for example, if all the characteristic numbers of the fundamental matrix of the solution are different from each other. In this case, as mentioned above, this highest value of the sum of characteristic numbers is not greater than the number

$$\mu = \chi \left\{ \exp \int_{t_0}^{t} \sum_{s=1}^{n} p_{ss}(t)\, dt \right\}.$$

Thus, the following inequality is valid in the most general case for all systems of linear differential equations:

$$\sigma \leqslant \mu. \qquad (3.2.1)$$

If the system of linear differential equations is such that

$$\sigma = \mu, \qquad (3.2.2)$$

it is called a *regular system*.

It directly follows from this that the necessary and sufficient condition for the regularity of a system of linear differential equations is the existence of a fundamental matrix of solutions, whose corresponding characteristic numbers satisfy the equality

$$\sum_{k=1}^{n} \chi_k = \chi \left\{ \exp \int_{t_0}^{t} \sum_{s=1}^{n} p_{ss}(t)\, dt \right\}. \qquad (3.2.3)$$

Naturally, the problem of a direct establishment of the regularity of any system of linear differential equations is equivalent to solving this system, since the fundamental matrix of the solutions of the system of equations is assumed to be known and it is possible to determine the highest value of the sum of the corresponding characteristic numbers. In some cases, however, the regularity of the system of linear equations may be established by using the criteria of regularity, where the corresponding conditions are imposed only on the coefficients of the system of equations (3.3).

We shall consider some of these criteria.

Theorem 3.2.1 [13]. *The sufficient condition for the system of equations* (3.3) *to be regular is the existence of matrix* $F = \| f_{sk}(t) \|_n^n$ *which is bounded and continuous for* $t \geqslant t_0$, *whose inverse matrix* F^{-1} *and derivative matrix* $\dot{F}$ *are bounded, and which satisfies the following equalities for* $t \geqslant t_0$:

$$\sum_{j=1}^{n} \frac{\partial \det F}{\partial f_{kj}} \left(\sum_{l=1}^{n} f_{sl} p_{lj} + \dot{f}_{sj} \right) = 0 \quad (k > s = 1, \ldots, n-1). \tag{3.2.4}$$

In addition, the following limits exist for the matrix F:

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial \det F}{\partial f_{sj}} \left(\sum_{l=1}^{n} f_{sl} p_{lj} + \dot{f}_{sj} \right) dt / \det F \tag{3.2.5}$$

$$(s = 1, \ldots, n).$$

It should be noted that the condition of boundedness and continuity (for $t \geqslant t_0$) of the matrix F having a bounded inverse matrix F^{-1} and a derivative matrix $\dot{F}$ and satisfying the condition (3.2.4) was established in [34]. Hence (3.2.5) is the only significant condition of this theorem.

It should be also observed that in this quite general theorem, only $n(n-1)/2$ constraints (3.2.4) and (3.2.5) have so far been imposed on the n^2 elements of the matrix F. Hence, by somehow imposing additional restrictions on the coefficients of the system of equations (3.3), we can obtain from this theorem a number of sufficient criteria of the regularity of the system (3.3) of equations under consideration in different particular cases.

Theorem 3.2.2 [13]. *If the system of equations* (3.3) *is almost diagonal, the characteristic numbers of its solutions* (3.1.15) *are strict, and the following inequalities hold for* $t \geqslant t_0$:

$$p_{s-1,s-1}(t) \geqslant p_{ss}(t) + \varepsilon \quad (s = 2, \ldots, n), \tag{3.2.6}$$

where ε *is a constant number, then this system is called a regular system.*

Theorem 3.2.3 [27]. *The necessary and sufficient condition for the regularity of the triangular system of equations*

$$\dot{x}_s = p_{s1}(t)\, x_1 + \ldots + p_{ss}(t)\, x_s \quad (s = 1, \ldots, n) \tag{3.2.7}$$

is that the following limits should exist:

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} p_{ss}(t)\, dt \quad (s = 1, \ldots, n). \tag{3.2.8}$$

Theorem 3.2.4 [27]. *A system of linear differential equations with constant coefficients is regular.*

The last theorem allows us to indicate a fairly general class of regular systems of linear differential equations, called reducible systems. For this purpose, we represent the system of linear equations (3.3) in the vector-matrix form

$$\dot{x} = P(t)\, x, \tag{3.2.9}$$

where $x = \operatorname{colon} \| x_1, \ldots, x_n \|$, $P(t) = \| p_{sk}(t) \|_n^n$.

We can transform this system of equations with the help of the linear substitution

$$y = F(t)\, x, \tag{3.2.10}$$

where $y = \operatorname{colon} \| y_1, \ldots, y_n \|$, $F = \| f_{sk}(t) \|_n^n$.

If the matrices $F(t)$, $\dot{F}(t)$, and $\det F^{-1}(t)$ are continuous and bounded for $t \geqslant t_0$, the transform (3.2.10) is called *Lyapunov's transform*. It should be noted that this transformation reduces the system (3.2.9) to the system

$$\dot{y} = Q(t)\, y, \tag{3.2.11}$$

where $Q(t) = \dot{F}F^{-1} + FPF^{-1}$.

In this case, the characteristic numbers of the solutions of the system (3.2.9) and the transformed system (3.2.11) are identical and the regularity of the system (3.2.9) leads to the regularity of the system (3.2.11), and vice versa [27].

Let us suppose that there exists a Lyapunov transform (3.2.10) which reduces the system of linear equations (3.2.9) to a system with constant coefficients,

$$\dot{y} = Ay, \tag{3.2.12}$$

where $A = \| a_{sk} \|_n^n$ $(\dot{F}F^{-1} + FPF^{-1} = A)$. The system of equations (3.2.9) is then called *reducible*. It directly follows from the above-mentioned properties of a Lyapunov transform (invariance of the characteristic numbers and conservation of the regularity) that reducible systems of linear differential equations with variable coefficients form a regular system.

Let us consider the criterion for the reducibility of a system of differential equations. This criterion can be considered as one of the criteria of the regularity.

Theorem 3.2.5 [35]. *The necessary and sufficient condition for the reducibility of the system* (3.3) *is the existence of a fundamental matrix of the solutions of the adjoint system*

$$\dot{z} + zP(t) = 0 \tag{3.2.13}$$

in the form

$$Z = e^{-At}F(t), \tag{3.2.14}$$

where $A = \| a_{sk} \|_n^n$ *is a constant matrix, and* $F = \| f_{sk}(t) \|_n^n$ *is a continuous and bounded matrix for* $t \geqslant t_0$ *with a bounded determinant* det F^{-1}.

It should be observed that if the following equality holds for all $t \geqslant t_0$:

$$\int_{t_0}^{t} \sum_{s=1}^{n} p_{ss}(t)\,dt = t \sum_{s=1}^{n} a_{ss} + \varepsilon(t), \tag{3.2.15}$$

where $\varepsilon(t)$ is a bounded function, then det F^{-1} is also bounded for $t \geqslant t_0$ [35].

3.3. Stability Theorems. Consider a nonlinear nonstationary material system whose perturbed motion is described by Eqs. (1.12).

The following theorem, called *Lyapunov's theorem* [27], allows us to establish the stability of the transient unperturbed motion $x = 0$ of the system under consideration, on the basis of the corresponding first-approximation equations (1.13).

Theorem 3.3.1. *If the system* (1.13) *of differential equations in the first approximation is regular, and the smallest characteristic number of its solutions is positive, then the unperturbed motion* $x = 0$ *is stable when we also take into account the higher-order terms of the system of equations* (1.12) *of the perturbed motion.*

The proof of this theorem, proposed by N.G. Chetaev [28], established that under the same conditions of the theorem, the unperturbed motion $x = 0$ will not only be asymptotically stable, but will also have positive characteristic numbers of the solutions of the equations (1.12) of perturbed motion. The latter is a very useful correction, for example, for investigating the stability of motion with respect to the coordinates ($Q = q$) when the stability with respect to velocities ($Q = \dot{q}$) has been established with the help of this very theorem.

Let us consider a nonlinear stationary system. In this case, the characteristic numbers of the solutions of the equations in the first approximation are defined as the real parts of the roots of the corresponding characteristic equation

$$D(p) \equiv (-1)^n \det \| p_{sk} - \delta_{sk} p \|_n^n = 0, \qquad (3.3.1)$$

taken with the opposite signs, while the system of equations in the first approximation is itself regular.

Hence, the stability of a steady-state unperturbed motion may be established with the help of the following Lyapunov theorem [27].

Theorem 3.3.2. *If the real parts of all the roots of the characteristic equation* (3.3.1) *are negative, the unperturbed steady-state motion* $x = 0$ *is asymptotically stable when we take into account the higher-order terms in the equations of the perturbed motion.*

Thus, the sufficient condition for stability in the case under consideration is that the roots of the characteristic equation (3.3.1) should lie on the left of the imaginary axis in the plane of the roots $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}$.

The characteristic equation (3.3.1) can be represented in the following form:

$$D(p) \equiv p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n = 0, \qquad (3.3.2)$$

where the coefficients $a_1, \ldots, a_n$ are the known expressions which depend on the coefficients $p_{sk} = \text{const}$ of the equations of perturbed motion in the first approximation.

We form the following matrix with the coefficients of Eq. (3.3.2):

$$H = \begin{Vmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ 1 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & a_n \end{Vmatrix}. \tag{3.3.3}$$

The diagonal determinants of this matrix,

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{Vmatrix} a_1 & a_3 \\ 1 & a_2 \end{Vmatrix}, \quad \ldots, \quad \Delta_n = a_n \Delta_{n-1}, \tag{3.3.4}$$

are called the *Hurwitz determinants*.

Using the well-known criteria of placing the roots of Eq. (3.3.2) to the left of the imaginary axis in the plane of the roots, we can formulate the following theorems on the stability in the first approximation, when the system under consideration is stationary [36, 37].

Theorem 3.3.3. *The unperturbed motion $x = 0$ is asymptotically stable if all the Hurwitz determinants* (3.3.4) *of the corresponding characteristic equation* (3.3.2) *are positive.*

Theorem 3.3.4. *The unperturbed motion $x = 0$ is asymptotically stable if all the coefficients of the characteristic equation* (3.3.2) *are positive and satisfy the conditions*

$$\Delta_3 > 0, \quad \Delta_5 > 0, \quad \Delta_7 > 0, \ \ldots \tag{3.3.5}$$

or

$$\Delta_2 > 0, \quad \Delta_4 > 0, \quad \Delta_6 > 0, \ \ldots. \tag{3.3.6}$$

It should be noted that these theorems on the stability in the first approximation are also valid when the system is not stationary, but such that only the coefficients of the second- and higher-order terms in the equations of perturbed motion contain time explicitly.

3.4. Solution of the Problems of Constructing Stable Systems. The method of characteristic numbers is quite effective for solving the above-mentioned problem (Sec. 2, Ch. 2) of constructing stable material systems, when

the motions of the system under consideration are described by the equations

$$\mathcal{L}(q, \dot{q}, \ddot{q}, t, u) = 0 \tag{3.4.1}$$

and it is required to determine the control vector $u(t)$ (parameters of the system and the generalized controlling forces) such that the given motion

$$\Omega\colon\ q = \varphi(t), \quad \varphi(t) \in C_t^2\ \{t \geqslant t_0\} \tag{3.4.2}$$

is one of the possible motions of the system and is stable with respect to the given indices of motion (the reference functions)

$$Q\,[Q_1, \ldots, Q_n],\ Q_\nu(q, \dot{q}, t) \in C^1 \quad (\nu = 1, \ldots, n). \tag{3.4.3}$$

In order to solve this problem, we form first of all the necessary conditions for realizing the given motion

$$\mathcal{L}(\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), t, u) = 0. \tag{3.4.4}$$

After this, the given motion (3.4.2) is treated as the unperturbed motion of the system and equations for the perturbed motion are constructed to the first approximation:

$$\dot{x}_s = p_{s1}(t, u)\,x_1 + \ldots + p_{sn}(t, u)\,x_n \quad (s = 1, \ldots, n). \tag{3.4.5}$$

In this case it is necessary to ensure that the higher-order terms in the equations of perturbed motion satisfy the conditions mentioned while formulating the stability problem in the first approximation (Sec. 3, Ch. 2).

With the help of the above-mentioned theorems on regularity and on characteristic numbers (Secs. 3.1 and 3.2, Ch. 2), we then construct sufficient conditions for the regularity of the system (3.4.5) of the equations obtained above, and for the positiveness of the characteristic numbers of its solutions. These conditions can be expressed in terms of the following inequalities:

$$S(\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), t, u) > 0. \tag{3.4.6}$$

In view of Lyapunov's theorem on the stability in the first approximation, these conditions will be the sufficient conditions for the stability of the given motion.

The required parameters of the system and the controlling forces $u_1(t), \ldots, u_m(t)$ will be determined as bounded and continuous components (for $t \geqslant t_0$) of the vector function $u(t)$, which satisfies the conditions (3.4.4), as well as the inequalities (3.4.6).

Let us consider the problem of constructing stable systems, when the perturbed motions of the system are described by the equations

$$\dot{x}_s = p_{s1}(u)\, x_1 + \ldots + p_{sn}(u)\, x_n + X_s^{(2)}(x, t, u) \qquad (s = 1, \ldots, n), \tag{3.4.7}$$

where $X^{(2)}(x, t, u)$ are holomorphic functions in the domain $H\,\{\sum_{s=1}^{n} x_s^2 \leqslant H\} \times T\,\{t \geqslant t_0\} \times U\,\{u\}$, whose expansions in powers of $x_1, \ldots, x_n$ begin with terms of not lower than second order, and when it is required to determine the value of the vector of the constant parameters of the system $u\,[u_1, \ldots, u_m]$, such that the unperturbed motion $x = 0$ is stable.

In the case under consideration, the coefficients p_{sk} in the first-approximation equations are constant, and the solution of the problem is reduced, in accordance with the Lyapunov stability theorem in the first approximation, to finding such points $M(u_1, \ldots, u_m)$ in the space of the parameters $\{u\}$, for which the roots of the equation

$$D(p, u) \equiv (-1)^n \det \| p_{sk}(u) - \delta_{sk} p \|_n^n = 0, \tag{3.4.8}$$

which can be expressed in the form

$$D(p, u) \equiv p^n + a_1(u)\, p^{n-1} + \ldots + a_{n-1}(u)\, p + a_n(u) = 0, \tag{3.4.9}$$

lie to the left of the imaginary axis in the plane of the roots $\{\operatorname{Re} p, \operatorname{Im} p\}$.

The set of points $M(u_1, \ldots, u_m)$ having the above-mentioned property is called the *stability domain of the space of the parameters* $\{u\}$, or the $D\,\{n, 0\}$ *domain.*

We shall designate as $D\,\{n - m, m\}$ the domain in the space of parameters where the coordinates of the points

$M(u_1, \ldots, u_m)$ cause the positioning of m roots of Eq. (3.4.9) to the right of the imaginary axis in the plane of the roots, and $n - m$ roots to the left of it.

The isolation of D-domains in the space of the parameters, corresponding to different distributions of the roots of the equations (3.4.9) with respect to the imaginary axis in the plane of roots is called the *D-section* of the space of parameters. The hypersurfaces bounding these domains are called the *boundaries of the D-domains* [38].

Thus, a solution of the problem considered above is reduced to the construction of the boundaries of D-domains or, to be more precise, to the construction of the domain of stability $D\{n, 0\}$ in the space of the parameters $\{u\}$.

It should be noted that it may turn out while solving specific problems that some of the domains $D\{n, 0\}$, $D\{n-1, 1\}, \ldots, D\{0, n\}$ do not exist, including the stability domain $D\{n, 0\}$. In the latter case, the system under consideration is structurally unstable, and it is then impossible to attain stability by simply choosing the values of the parameters of the system.

Different methods have been worked out for constructing the boundaries of the domain of stability $D\{n, 0\}$ [38]. Each of these methods is a direct consequence of some criterion of positioning the roots of an algebraic equation to the left of the imaginary axis of the plane of the roots.

We shall describe some of these methods.

1. The Hurwitz determinants Δ_i $(i = 1, \ldots, n)$ are constructed for the equation (3.4.9). In general, all these determinants are functions of the required parameters $u_1, \ldots, u_m$.

After this, we form the inequalities

$$\Delta_i(u_1, \ldots, u_m) > 0 \quad (i = 1, \ldots, n) \qquad (3.4.10)$$

for these parameters, and thus isolate the domain of values of these parameters, in which the stability of the unperturbed motion $x = 0$ is ensured.

2. If the coefficients of the characteristic equation (3.4.9) depend only on one parameter u_0, the curves $\Delta_i = \Delta_i(u_0)$ $(i = 1, \ldots, n)$ are plotted in the plane $\{u_0, \Delta_i\}$. In this case, the segment of the abscissa axis, where all the curves $\Delta_i = \Delta_i(u_0)$ are situated in the upper half-plane, isolates the required domain of stability.

3. In the general case, the boundaries of the D-domains are defined by the equations

$$\begin{aligned} a_n (u_1, \ldots, u_m) &= 0, \\ \Delta_{n-1} (u_1, \ldots, u_m) &= 0. \end{aligned} \tag{3.4.11}$$

These equations, which are also the conditions that at least one or two roots are situated on the imaginary axis, isolate different D-domains in the space, including the domain of stability $D\ \{n, 0\}$ (if it does exist) which must be determined by establishing the distribution of roots at the test points of these domains.

4. The boundaries of D-domains are also described by the equations

$$\operatorname{Re} D (i\omega, u) = 0, \quad \operatorname{Im} D (i\omega, u) = 0 \tag{3.4.12}$$

when ω changes from $-\infty$ to $+\infty$. These equations, mapping the imaginary axis in the plane of the roots of Eq. (3.4.9) onto the boundaries of D-domains and vice versa, also isolate the D-domains in the space of the parameters.

It should be noted that the method of constructing the D-domains with the help of Eqs. (3.4.11) or (3.4.12) should be applied in the case when the number of parameters to be determined has been reduced to two. In this case, the boundaries of the D-domains are plane curves which can be actually constructed, and even the domain of stability itself may be found quite easily.

Sec. 4. The Ascending Motion of a Point with Varying Mass in Gravitational Field

As an example of the application of the method of characteristic numbers for constructing stable systems, let us consider the solution of the following inverse problem of the dynamics of a point with varying mass [13]:

Find the law of variation of mass, $m(t)$, and the velocity $\mathbf{u}(t)$ of escaping mass of a point with varying mass in the gravitational field, such that the given motion in the vertical plane

$$\begin{aligned} &y = \varphi (t), \\ &z = \psi (t) \quad (\lim_{t \to \infty} \psi (t) = \infty, \ \dot{\varphi} (t) \neq 0, \ \dot{\psi} (t) \neq 0) \end{aligned} \tag{4.1}$$

is one of the possible motions of this point, and is stable.

The corresponding equations of motion of a point of varying mass under a quadratic resistance law constitute the following system [7]:

$$\begin{aligned} m\ddot{y} &= \dot{m}(\mu - 1)\dot{y} - k_1 v\dot{y} - k_2 v\dot{z}, \\ m\ddot{z} &= \dot{m}(\eta - 1)\dot{z} - k_1 v\dot{z} + k_2 v\dot{y} - mg, \end{aligned} \tag{4.2}$$

where $\mu = \mu(t)$ and $\eta = \eta(t)$ are the ratios of the projections of the velocity **u** of the escaping mass, and the velocity **v** of the point itself, onto the coordinate axes, $Y = k_1 v^2$ and $Z = k_2 v^2$ are the components of the resistance forces along the tangent and the normal to the trajectory respectively, $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are the correction factors in the quadratic law for forces, which satisfy the conditions $\lim\limits_{t\to\infty} k_1(t) = 0$ and $\lim\limits_{t\to\infty} k_2(t) = 0$, respectively.

The necessary conditions for realizing the given motion (4.1) are given by

$$\begin{aligned} \frac{\dot{m}}{m}(\mu - 1) &= \frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_1\dot{\varphi} + k_2\dot{\psi}}{m\dot{\varphi}} v_0, \\ \frac{\dot{m}}{m}(\eta - 1) &= \frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} + \frac{k_1\dot{\psi} - k_2\dot{\varphi}}{m\dot{\psi}} v_0, \end{aligned} \tag{4.3}$$

where $v_0 = \sqrt{\dot{\varphi}^2 + \dot{\psi}^2}$ is the velocity of a point in the given motion (4.1).

It follows from the above that if the functions $\varphi(t)$ and $\psi(t)$ are specified, the required parameters μ and η obey only two conditions. This allows us to impose an additional condition on these parameters, which in the present case should be the condition of stability of the given motion (4.1).

Let us determine the conditions under which the given motion (4.1) of a point with varying mass in the gravitational field will be stable with respect to the coordinates and velocities.

Considering the given motion of the point to be unperturbed, we form the equations of the perturbed motion:

$$
\begin{aligned}
\dot{x}_1 &= \left(\frac{\ddot{\varphi}}{\dot{\varphi}} - \frac{k_1\dot{\varphi}^3 - k_2\dot{\psi}^3}{mv_0\dot{\varphi}}\right) x_1 - \frac{k_1\dot{\varphi}\dot{\psi} + k_2\dot{\varphi}^2 + 2k_2\dot{\psi}^2}{mv_0} x_2 \\
&\qquad + X_1^{(2)}(x_1, x_2, t), \\
\dot{x}_2 &= -\frac{k_1\dot{\varphi}\dot{\psi} - k_2\dot{\psi}^2 - 2k_2\dot{\varphi}^2}{mv_0} x_1 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_1\dot{\psi}^3 + k_2\dot{\varphi}^3}{mv_0\dot{\psi}}\right) x_2 \\
&\qquad + X_2^{(2)}(x_1, x_2, t), \\
\dot{x}_3 &= x_1, \\
\dot{x}_4 &= x_2,
\end{aligned}
\tag{4.4}
$$

where x_1 and x_2 are the perturbations in the projections of the velocity of the point onto the coordinate axes, x_3 and x_4 are the perturbations of the coordinates of the point, and $X_1^{(2)}$ and $X_2^{(2)}$ are functions whose expansions in powers of x_1 and x_2 begin with terms of not lower than the second order.

Henceforth, we shall assume that the coefficients of different powers of x_1 and x_2 in Eqs. (4.4) are bounded and continuous for $t \geqslant t_0$. For this purpose, it is sufficient to require that the given condition is satisfied by the functions

$$
\ddot{\varphi}, \quad \ddot{\psi}, \quad \frac{k_i}{m}\dot{\varphi}^{n+1-\nu}\dot{\psi}^{\nu}, \quad \frac{k_i}{m}\dot{\varphi}^{\nu}\dot{\psi}^{n+1-\nu} \qquad (i = 1, 2)
$$

for all natural numbers n and ν.

Let us consider the first-approximation equation for the perturbations in the projections of the velocity of a point:

$$
\begin{aligned}
\dot{x}_1 &= \left(\frac{\ddot{\varphi}}{\dot{\varphi}} - \frac{k_1\dot{\varphi}^3 - k_2\dot{\psi}^3}{mv_0\dot{\varphi}}\right) x_1 - \frac{k_1\dot{\varphi}\dot{\psi} + k_2\dot{\varphi}^2 + 2k_2\dot{\psi}^2}{mv_0} x_2, \\
\dot{x}_2 &= -\frac{k_1\dot{\varphi}\dot{\psi} - k_2\dot{\psi}^2 - 2k_2\dot{\varphi}^2}{mv_0} x_1 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_1\dot{\psi}^3 + k_2\dot{\varphi}^3}{mv_0\dot{\psi}}\right) x_2.
\end{aligned}
\tag{4.5}
$$

In view of the assumption concerning the coefficients k_1 and k_2, the system of equations (4.5) is nearly diagonal.

Hence, by using Theorem 3.1.4 on the characteristic numbers (Sec. 3, Ch. 2), we find that if the inequality

$$\frac{\ddot{\varphi}}{\dot{\varphi}}-\frac{\ddot{\psi}}{\dot{\psi}}-\frac{g}{\dot{\psi}}+\frac{k_1(\dot{\psi}^2-\dot{\varphi}^2)\dot{\varphi}\dot{\psi}+k_2(\dot{\varphi}^4+\dot{\psi}^4)}{mv_0\dot{\varphi}\dot{\psi}}>\varepsilon>0$$

is satisfied for $t \geqslant t_0$, or, in other words, in view of the conditions (4.3), if

$$k_1(\dot{\psi}^2-\dot{\varphi}^2)-2k_2\dot{\varphi}\dot{\psi}+\dot{m}(\mu-\eta)v_0>\varepsilon>0, \tag{4.6}$$

where ε is an indefinitely small number, the characteristic numbers of the solutions of the system of equations (4.5) are given by the following expressions:

$$\begin{aligned}\chi_1 &= -\overline{\lim_{t\to\infty}}\frac{1}{t}\left(\ln\dot{\varphi}-\int_{t_0}^{t}\frac{k_1\dot{\varphi}^3-k_2\dot{\psi}^3}{mv_0\dot{\varphi}}\,dt\right),\\ \chi_2 &= -\overline{\lim_{t\to\infty}}\frac{1}{t}\left(\ln\dot{\psi}-\int_{t_0}^{t}\frac{k_1\dot{\psi}^3+k_2\dot{\varphi}^3}{mv_0\dot{\psi}}\,dt+g\int_{t_0}^{t}\frac{dt}{\dot{\psi}}\right).\end{aligned} \tag{4.7}$$

Moreover, in view of the inequality (4.6), we get

$$\chi_1<\chi_2. \tag{4.8}$$

Suppose that the characteristic numbers χ_1 and χ_2 are strict. Then the system of equations (4.5) is a regular system. This can be confirmed by directly establishing the validity of the equality

$$\chi_1+\chi_2=\chi\left\{\exp\int_{t_0}^{t}\mathrm{Tr}\,P\,dt\right\},$$

where Tr P is the trace of the matrix of the coefficients of the system of equations (4.5).

Consequently, if besides the condition (4.6) we require that the characteristic numbers of the solutions of the system of equations (4.5) be positive, which ensures that the equality

$$\lim_{t\to\infty}\frac{\ln\dot{\varphi}}{t}<\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\frac{k_1\dot{\varphi}^3-k_2\dot{\psi}^3}{mv_0\dot{\varphi}}\,dt \tag{4.9}$$

is satisfied, then in view of Lyapunov's theorem 3.3.1 (Sec. 3, Ch. 2) the given motion (4.1) is stable with respect to the projections of the velocity of a point. Moreover, this stability is asymptotic and has positive characteristic numbers of perturbations of the projections of the velocity of the point. This means that the motion will be stable with respect to the coordinates also in view of the equations

$$\dot{x}_3 = x_1, \quad \dot{x}_4 = x_2. \tag{4.10}$$

It follows from above that by choosing the laws of variation of mass and velocity of the escaping mass, in view of the conditions (4.3), (4.6), and (4.9), the given motion (4.1) will be one of the possible motions of a point of varying mass in the gravitational field, and will be stable with respect to the coordinates and the velocity.

Sec. 5. Application of Lyapunov's Functions

Suppose that the equations of perturbed motion, obtained while solving the problem of constructing stable systems, are represented in the form

$$\dot{x}_s = X_s(x, t, u) \quad (X_s(0, t, u) = 0; \ s = 1, \ldots, n), \tag{5.1}$$

where $X_s(x, t, u)$ are holomorphic functions of the variables $x_1, \ldots, x_n$ in the domain $H \{ \sum_{s=1}^{n} x_s^2 \leqslant H\} \times T \{t \geqslant t_0\} \times \times U \{u\}$.

After this, the problem consists in the determination of the control vector $u\,[u_1, \ldots, u_m]$ such that the unperturbed motion $x = 0$, which is the given motion of the system, is a stable motion of the system.

In order to solve this problem, we make use of the method of Lyapunov's functions.

For this purpose, we first consider the necessary definitions and theorems on the stability of a nonlinear unstable system when the perturbed motions are described by the equations

$$\dot{x}_s = X_s(x, t) \quad (X_s(0, t) = 0; \ s = 1, \ldots, n), \tag{5.2}$$

where the functions $X_s(x, t)$ are assumed to be holomorphic in the variables $x_1, \ldots, x_n$ in the domain

$$H\{\sum_{s=1}^{n} x_s^2 \leqslant H\} \times T\{t \geqslant t_0\}. \qquad (5.3)$$

5.1. Lyapunov's Functions. Consider the functions $V(x, t)$, defined in the domain (5.3) and vanishing for the unperturbed motion $x = 0$. Suppose that these functions are single-valued, bounded, and continuous in their domain. Let us consider the following fundamental definitions of the classical theory of stability, concerning the method applied for constructing stable systems.

1. If the function $V(x, t)$, satisfying the given conditions for a sufficiently large t_0 and a sufficiently small H, assumes, besides zero values, the values having the same sign, this function is called a *sign-constant function*: *it is positive-constant for* $V(x, t) \geqslant 0$, *and negative-constant for* $V(x, t) \leqslant 0$.

2. If the function $W(x)$, which does not depend on t, is such that it vanishes in the domain $H\{\sum_{s=1}^{n} x_s^2 \leqslant H\}$ only for the unperturbed motion $x = 0$, it is called a *sign-definite function*: *it is positive-definite if* $W(x) > 0$, *and negative-definite if* $W(x) < 0$.

3. The function $V(x, t)$, which depends on t, is called *positive-definite* if we can find a positive-definite function $W(x)$, which does not depend on t, such that $V(x, t) \geqslant W(x)$ in the domain (5.3). If $-V(x, t) \geqslant W(x)$, the corresponding function $V(x, t)$ is called *negative-definite.*

4. If the function $V(x, t)$ is such that for any positive number l, however small, there exists a number λ so that the inequality $|V(x, t)| < l$ holds for all $t \geqslant t_0$ and $\sum_{s=1}^{n} x_s^2 \leqslant \lambda$, the function is said to *have an infinitely small upper limit.*

It should be noted that if the function $V(x, t)$ is positive-definite, the surface $V(x, t) = l$ in the domain (5.3) is a closed surface, deforming in time, and covering the origin of coordinates. This function remains for all $t \geqslant t_0$ within the stationary surface $W(x) = l$, which is also closed, if

$V(x, t) \geqslant W(x) > 0$ at all points of the domain (5.3) except the origin of the coordinates.

If the positive-definite function $V(x, t)$ also has an infinitely small upper limit, all the points of the closed surface $V(x, t) = l$ lie outside the sphere $\sum_{s=1}^{n} x_s^2 = \lambda$ corresponding to the number l.

Thus, the condition that the function $V(x, t)$ be sign-definite is geometrically interpreted as a definite restriction on the displacements of the points on the surface $V(x, t) = l$ in the direction away from the origin of coordinates with the passage of time, while the condition of the existence of an infinitely small limit is considered as a restriction on the displacements of the points of this very surface inwards, i.e. towards the origin of coordinates.

The functions $V(x, t)$ having any of the above-mentioned properties (constant sign, definite sign, infinitely small limit) are called *Lyapunov's functions.*

Henceforth, we shall also assume that Lyapunov's functions are differentiable in the domain (5.3) in such a way that we can also draw conclusions about the constant sign or the definite sign concerning their time derivatives which are obtained from the equations (5.2) of the perturbed motion:

$$\dot{V}(x, t) = \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s(x, t) + \frac{\partial V}{\partial t}, \qquad (5.1.1)$$

5.2. Stability Theorems. Let us consider a nonlinear unstationary material system whose perturbed motions are described by the Eqs. (5.2).

The following Lyapunov's theorems [27] can be used to establish the stability of the transient unperturbed motion $x = 0$.

Theorem 5.2.1. *If the differential equations* (5.2) *of the perturbed motion are such that we can find a sign-definite function $V(x, t)$ whose derivative $\dot{V}(x, t)$, in view of these conditions, is either a sign-constant function with a sign opposite to that of $V(x, t)$, or is identically equal to zero, the unperturbed motion $x = 0$ is stable.*

Theorem 5.2.2. *If for the system of equations* (5.2) *of perturbed motion there exists a sign-definite function* $V(x, t)$ *having an infinitely small upper limit and a derivative* $\dot{V}(x, t)$, *formed according to this system of equations, which is a sign-definite function with a sign opposite to that of* $V(x, t)$, *then the perturbed motion is asymptotically stable.*

It should be noted that the method of Lyapunov's functions can be used to draw conclusions about stability even from the first approximation equations

$$\dot{x}_s = p_{s1}(t)\,x_1 + \ldots + p_{sn}(t)\,x_n \quad (s = 1, \ldots, n) \quad (5.2.1)$$

of the system of equations (5.2) for the perturbed motion. In this case, the stability of the unperturbed motion may be established with the help of Malkin's theorem [30].

Theorem 5.2.3. *If for the first approximation equations* (5.2.1) *there exists a sign-definite function* $V(x, t)$ *having an infinitely small upper limit and a sign-definite derivative* $\dot{V}(x, t)$ *in accordance with these equations (with a sign opposite to that of* $V(x, t)$*), the unperturbed motion* $x = 0$ *is asymptotically stable even when we take into account higher-order terms* $X_s^{(2)}(x, t)$ *of the system of equations* (5.2) *of the perturbed motion if the conditions*

$$|X_s^{(2)}(x, t)| < A(|x_1| + \ldots + |x_n|) \quad (5.2.2)$$

are satisfied for a sufficiently small constant A, *in the entire domain of the right-hand sides of the system of equations* (5.2) *of the perturbed motion.*

It should be also noted that the theorems enunciated above are also valid in the case of nonlinear stationary systems and can be used for determining the constant values of components of the control vector u in the problem of constructing stable stationary systems.

Thus, the problem of establishing the stability with the help of Lyapunov's functions is reduced to the construction of appropriate functions $V(x, t)$, which satisfy the requirements of any of the above theorems.

5.3. Solution of the Problem of Constructing Stable Systems. Suppose that the first part of the above-mentioned problem of constructing stable systems (Sec. 2, Ch. 2) has

been solved, i.e. the necessary conditions for realizing the given motion

$$\Omega: q = \varphi(t), \quad \varphi(t) \in C^2 \{t \geqslant t_0\} \tag{5.3.1}$$

have been defined in the form of the equalities

$$\mathscr{L}(\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), t, u) = 0. \tag{5.3.2}$$

Further, we assume that these conditions define the control vector $u(t)$ with a certain freedom in the choice of its components.

Beyond this, the problem consists in determining the components $u_1(t), \ldots, u_m(t)$ of the control vector in such a way that the given motion (5.3.1) is stable (asymptotically stable) with respect to the given qualitative indices of motion $Q_s(q, \dot{q}, t)$ $(s = 1, \ldots, n)$ when they are initially perturbed.

In order to solve this problem, we make use of the method of Lyapunov's functions. For this purpose, we first form the appropriate system of equations of the perturbed motion

$$\dot{x}_s = X_s(x, t, u) \quad (s = 1, \ldots, n), \tag{5.3.3}$$

whose trivial solution $x = 0$ corresponds to the given motion (5.3.1) of the system.

Then, with a view to apply any of the above-mentioned theorems on stability, we form a certain positive-definite function $V(x, t)$. The time derivative $\dot{V}(x, t)$ of this function is determined in accordance with the equations (5.3.3) of the perturbed motion.

The sufficient conditions for stability are defined, in accordance with Lyapunov's theorem 5.2.1, as the conditions that this derivative be either negative constant or identically equal to zero.

The conditions corresponding to the requirement of asymptotic stability must be found as the conditions that the positive-definite function $V(x, t)$ constructed above has an infinitely small upper limit and has a negative-definite derivative in view of the equations (5.3.3) of the perturbed motion (Lyapunov's theorem 5.2.2).

Suppose that the conditions of stability (asymptotic stability) are represented in the form of the inequalities

$$S(\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), t, u) > 0. \qquad (5.3.4)$$

In this case, the components of the control vector $u(t)$ that solve the problem of constructing a stable system are obtained from these conditions (5.3.4) as bounded and continuous functions for $t \geqslant t_0$, satisfying at the same time the conditions (5.3.2) for realizing the given motion.

It should be noted that while solving the problem of constructing stable systems, we sometimes encounter difficulties in composing the very equations (5.3.3) of the perturbed motion. This is primarily due to the presence of unknown functions $u_1(t), \ldots, u_m(t)$ in the equations of motion. In such cases, we can confine ourselves to the construction of the equations of perturbed motion in the first approximation only, and to the determination of the stability conditions imposed on the control vector $u(t)$, with the help of Malkin's theorem 5.2.3 on stability in the first approximation.

It follows from the above that the most important step in the construction of stable systems with the help of Lyapunov's functions method is the choice of Lyapunov's function itself, satisfying the requirements of some stability theorem. These functions should be chosen in such a way that it should be possible to actually obtain easily interpretable stability conditions which could be physically realized.

The general concepts of Lyapunov's functions and certain methods of their construction are described in the well-known monographs [27-31, 39, 40]. The following methods are widely used for investigating the stability: the method of constructing Lyapunov's function in the form of a bunch of integrals of the equations of perturbed motion [28], the method of application of Lyapunov's functions, constructed for some simple systems, for analyzing more complex systems [41], the construction of Lyapunov's function as a sum of quadratic form of perturbations and a certain integral of nonlinear terms on the right-hand sides of the equations of perturbed motion [31], and so on. Naturally, all these methods can also be applied for solving the problem of constructing stable systems.

It should be observed that a simple and yet quite efficient method in several problems of constructing stable systems is to obtain Lyapunov's function as a quadratic form with varying coefficients (depending on t and u). The choice of these coefficients is dictated by the requirements of simplicity and the possibility of physical interpretation of the stability conditions expected in this case, as well as by the actual realization of these conditions.

Sec. 6. Permanent Rotation of a Self-excited Gyroscope

To begin with, let us consider the following inverse problem.

Find the moments $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ of exciting forces applied to a gyroscope, relative to the central principal axes x, y, z, and the conditions imposed on the moments of inertia A, B, and C of the gyroscope, relative to the same axes, such that the rotational motion of the gyroscope with the given properties

$$\Omega : \begin{cases} \omega_1 \equiv x_1 - a \exp\left\{-\frac{b}{A} t\right\} = c_1, \\ \omega_2 \equiv x_2^2 + x_3^2 = c_2, \end{cases} \tag{6.1}$$

where x_1, x_2, and x_3 are the projections of the instantaneous angular velocity onto the principal axes, and a, b, c_1, and c_2 are constants, is one of the possible motions of the gyroscope.

In order to solve this problem, we first formulate the necessary conditions for the realizability of the given motion. It is well known that these conditions are equations in the required quantities in solving the inverse problems of dynamics.

Taking into account the corresponding equations of motion of the gyroscope (Euler's equations)

$$\begin{aligned} A\dot{x}_1 &= (B - C)\, x_2 x_3 + \mathcal{L}_1, \\ B\dot{x}_2 &= (C - A)\, x_3 x_1 + \mathcal{L}_2, \\ C\dot{x}_3 &= (A - B)\, x_1 x_2 + \mathcal{L}_3, \end{aligned} \tag{6.2}$$

these conditions can be represented as follows:

$$(B-C)\,x_2x_3+\mathcal{L}_1+ab\exp\left\{-\frac{b}{A}\,t\right\}=0,$$

$$[C\,(C-A)-B\,(B-A)]\,x_1x_2x_3+C\mathcal{L}_2x_2+B\mathcal{L}_3x_3=0. \tag{6.3}$$

It follows hence that the motion of a gyroscope with the given properties (6.1) is possible if

$$\mathcal{L}_1=b\,(c_1-x_1),\quad \mathcal{L}_2=\mathcal{L}_3=0,\quad B=C. \tag{6.4}$$

Let us assume that the simplest possibility of realizing the given rotational motion (6.1) of the gyroscope is the required solution of this problem. In this case, the equations of motion of the gyroscope will be represented in the form

$$\begin{aligned} A\dot{x}_1 &= b\,(c_1-x_1),\\ B\dot{x}_2 &= (B-A)\,x_1x_3,\\ B\dot{x}_3 &= (A-B)\,x_1x_2. \end{aligned} \tag{6.5}$$

These equations have a trivial solution

$$x_1=c_1,\quad x_2=x_3=0 \tag{6.6}$$

corresponding to the permanent rotation of the gyroscope about its principal axis, i.e. the x-axis.

Let us apply the method of Lyapunov's functions for establishing the stability conditions of this permanent rotation of the gyroscope relative to the projections x_1, x_2, x_3 of the instantaneous angular velocity.

Considering the permanent rotation (6.6) as the unperturbed motion of the gyroscope, we construct the corresponding equations of the perturbed motion

$$\begin{aligned} A\dot{y}_1 &= -by_1,\\ B\dot{y}_2 &= (B-A)\,(y_1+c_1)\,y_3,\\ B\dot{y}_3 &= (A-B)\,(y_1+c_1)\,y_2, \end{aligned} \tag{6.7}$$

where y_1, y_2, y_3 are the perturbations of the projections of the instantaneous angular velocity of the gyroscope ($y_1=x_1-c_1$, $y_2=x_2$, $y_3=x_3$).

It should be noted that Lyapunov's method of characteristic numbers does not lead to a solution of the stability problem posed above, since for $b > 0$ the corresponding characteristic equation has a couple of purely imaginary roots (critical case).

Lyapunov's function in the problem under consideration may be constructed in the form

$$W = y_1^2 + y_2^2 + y_3^2.$$

The time derivative of this positive-definite function, calculated in accordance with the equations (6.7) of the perturbed motion,

$$\dot{W} = -2 \frac{b}{A} y_1^2$$

is a constant negative function of the variables y_1, y_2, y_3 for $b > 0$.

Hence, in view of Lyapunov's theorem 5.2.1 (Sec. 5, Ch. 2), the permanent rotation (6.6) of a self-excited symmetric gyroscope is stable under the condition that $b > 0$.

Sec. 7. Construction of Reference Functions

In the stability theory, the possible perturbed motions of a material system are compared with the unperturbed motions with respect to the corresponding values of the given kinematic indices of motion at every instant of time $t \geqslant t_0$. In this case, it is assumed that the kinematic indices of motion may be described by the vector function $Q(y, \dot{y})$, defined in a certain spatial domain $G\{y_1, \ldots, y_n\}$ of variation of the phase coordinates of the system under consideration. The investigation of stability of the unperturbed motion is reduced to a determination of the behaviour of the difference in the values of this vector function for the possible perturbed motions and the unperturbed motions, respectively, of the system:

$$x = Q_{\text{p.m}} - Q_{\text{un.m}} \tag{7.1}$$

for all $t \geqslant t_0$. In the problems concerning the construction of stable systems, the required system parameters and the additional controlling forces are also determined finally from the conditions imposed on this difference in accordance

with the accepted definition of stability. As has been already agreed (see Sec. 1, Ch. 2), the components of the vector function $Q(y, t)$ are called the *reference functions.*

In the accepted formulations of the stability problems, the reference functions are given, just like the unperturbed motion itself and the equations of motion of the material system under consideration (in the problems concerning the analysis of stability) or the structure of these equations (in the problems of constructing stable systems). Thus, the solution of stability problems is reduced to the determination of conditions of stability of a given motion of the system under consideration with respect to the given reference functions.

However, in several applied problems in the theory of stability, it may be useful to determine the sets of the reference functions themselves, with respect to which the given motion of the material system under consideration is stable. By way of an example, let us consider the following inverse problem which is directly connected with the construction of stable systems [39].

Given the law of motion

$$\Omega\colon\ y = \varphi(t), \qquad \varphi(t) \in C' \ \{t \geqslant t_0\}, \tag{7.2}$$

construct (1) *an appropriate set of equations of motion of a material system,*

$$\dot{y} = Y(y, t) \tag{7.3}$$

in the class of equations which allow, under the initial conditions $y\,|_{t=t_0} = \varphi(t_0)$ *for the existence of the unique solution* (7.2), *and* (2) *a set of n-dimensional vector functions* $Q(y)$, *holomorphic in a certain* ε*-neighbourhood* $\Omega_\varepsilon\ \{\|y - \varphi(t)\| < \varepsilon\}$ *of the integral manifold* Ω (7.2) *for all* $t \geqslant t_0$, *with respect to whose components the given motion is stable in Lyapunov's sense.*

The solution of this problem will finally define a set of kinematic indices of motion (conditions imposed on them), with respect to which the given motion (7.2) of the system under consideration is stable.

Let us consider the solution of this problem with the help of Lyapunov's method of characteristic numbers.

The set of equations of motion of a material system, for which the given motion (7.2) is one of the possible motions, may be represented in the form

$$\dot{y} = \dot{\varphi}(t) + \Phi(y, t), \tag{7.4}$$

where $\Phi(y, t)$ is a certain vector function, holomorphic in the domain Ω_ε for $t \geqslant t_0$, and satisfying the condition $\Phi(y, t)_{y=\varphi(t)} = 0$.

In the first approximation, the equations of perturbed motion constructed for components of the vector function $Q(y)$ can be written in the form

$$\dot{x}_i = \sum_{\varkappa, \nu, s}^{1, n} \frac{A_\nu^\varkappa}{\Delta} \left(\frac{\partial \psi_\nu^i}{\partial \varphi_s} \dot{\varphi}_s + p_\nu^s \psi_s^i \right) x_\varkappa \quad (i = 1, \ldots, n), \tag{7.5}$$

where $p_\nu^s = p_\nu^s(t)$ are the coefficients of linear terms in the expansion of the function $\Phi_s(y, t)$ into a power series in ω_ν $(\nu = 1, \ldots, n)$; $\omega_\nu = y_\nu - \varphi_\nu(t)$ are the perturbations in the phase coordinates $(\nu = 1, \ldots, n)$; $\Delta = \det \mathscr{L}$, $\mathscr{L} = \| \psi_\nu^i \|_n^n$, $\psi_\nu^i = \left.\frac{\partial Q_i(y)}{\partial y_\nu}\right|_{y=\varphi(t)}$, and $A_\nu^\varkappa$ is the cofactor of the element $(\varkappa, \nu)$ of the determinant Δ.

With the help of the linear transformation

$$x_\varkappa = \sum_{s=1}^{n} \psi_s^\varkappa \omega_s \quad (\varkappa = 1, \ldots, n) \tag{7.6}$$

the system of equations (7.5) acquires the form

$$\dot{\omega}_s = \sum_{\varkappa=1}^{n} p_\varkappa^s \omega_\varkappa \quad (s = 1, \ldots, n). \tag{7.7}$$

Let us require that the system (7.7) of linear differential equations be regular and that the characteristic numbers of its solutions be positive. In this case, if (7.6) is a Lyapunov transform, the given motion (7.2) is stable with respect to the kinematic indices of motion, represented by the components of the vector function $Q(y)$.

Thus, if the equations of motion (7.2) of a material system are constructed in such a way that the system (7.7) of linear equations is regular and the characteristic numbers of its solutions are positive, the set of the required components of the

vector function $Q(y)$ may be determined from the following conditions:

$$\left.\begin{array}{l}\mathscr{L},\ \dot{\mathscr{L}} \text{ are bounded} \\ \det \mathscr{L}^{-1} \neq 0\end{array}\right\} \text{ for all } t \geqslant t_0. \qquad (7.8)$$

It should be noted that in this method of solving the problem formulated above, the stability conditions are directly imposed only on the coefficients p_ν^s of the linear in ω_ν terms of the function $\Phi_s(y, t)$ $(\nu, s = 1, \ldots, n)$.

It should be also noted that the system of equations (7.5) of the perturbed motion in the first approximation can also be used for solving the problem of constructing the equations of motion of a material system, whose given motion (7.2) is stable with respect to the components of the given vector function $Q(y)$. For this purpose, the coefficients p_ν^s (defining the linear part of the functions $\Phi_s(y, t)$ (linear with respect to ω_ν) are obtained with the help of the known stability conditions in the first approximation as applied to this system of linear differential equations. In this case, we must require that the conditions (7.8) be satisfied.

Chapter Three

CONSTRUCTION OF PROGRAMMED MOTION SYSTEMS

In the theory of controlled motion, the problems of analytic construction of material systems of different physical nature and different structure are solved in such a way that the process occurring in these systems satisfies some preset requirements. These requirements are in the form of the condition that a process with specified properties is realized in the system. These properties are called the *programme of the motion*, each individual property is called an *element of the programme*, and the corresponding process taking place in the system is called the *programmed motion* of the system.

The programmed motion of material systems is eventually realized by the action of additional controlling forces on the system. These forces are applied to the system from outside by other systems (for open control systems), are caused by corresponding changes in the parameters of the system (in the case of autonomous control systems), or are formed with the help of special control mechanisms which are included in the system under consideration (in the case of closed control systems). The problem of the *analytic construction of programmed motion systems* involves the determination of forces applied to a system in order to realize a programmed motion, the determination of the laws of variation of the system parameters, corresponding to a given programme, and the construction of the equations of motion of controlling equipment (regulators). It should be observed that the programmed motion must also be stable, especially with respect to the properties of motion themselves, if their initial values are different from their given values. Hence, the problems of the analytic construction of programmed motion systems include the problems of establishing the realizability of the programme itself, as well as the problems of determining the sufficient conditions for the stability of the programmed motion.

In the literature [13-22, 31, 42], a fairly detailed account is given of the formulation of the problems of analytic construction of programmed motion systems, a discussion of their results, as well as a number of problems related to the programmed motion of material systems. Here, we shall confine ourselves to a consideration of material systems whose motion is described by ordinary differential equations. The problem of the analytic construction of programmed motion systems can then be reduced in all the cases under consideration to corresponding inverse problems of dynamics, formulated with the additional requirement that the given motions (programmes of motion) be stable in Lyapunov's sense (if only initial deviations are present).

In this chapter, we shall formulate the problems of analytic construction of programmed motion systems from this point of view only. Such a treatment helps in reducing the problems of analytic construction of programmed motion systems to the construction of the corresponding equations of motion of these systems in accordance with the given particular integrals of the equations of motion in such a way that these integrals, which reflect the given properties of motion of the system under consideration, are stable.

We shall also formulate and solve the problem of constructing integral functionals which assume a stationary value for the solutions of the equations of motion constructed in accordance with the given programme. A solution of this problem with the subsequent physical interpretation of the functionals thus constructed allows us to determine a large number of the dynamic indices of programmed motion, which acquire stationary values on the programmed motion of the system under consideration. The integral dynamic indices of motion obtained in this way may serve as the initial functionals for an eventual formulation of the problems of analytic construction of the systems of optimal programmed motion.

Sec. 1. Formulation of the Problems of Constructing Programmed Motion Systems

The programmed motion of material systems may be realized directly by the action of additional controlling forces on the system, by an appropriate change in the parameters of the system during motion, by using special control devices in

the system, or by a suitable combination of these methods [13-15].

The analytic construction of programmed motion systems in these cases is reduced to the construction of such equations of motion of the material system under consideration, for which the given programmed motion is one of the possible motions. From the equations of motion constructed in this way, we can later determine the controlling forces as well as the corresponding parameters of the system which are necessary for realizing a given programmed motion of the system under consideration, and the equations of motion of the control devices, which close the equations of motion known a priori for the system.

However, the control elements obtained in this way (forces, parameters, and regulators) cannot ensure an exact accomplishment of the programmed motion of a system. Programmed motion will take place only when the initial values of the generalized coordinates and velocities of the real motion coincide with their values for the programmed motion, i.e. when there are no initial perturbations. In actual practice, however, these initial perturbations always take place, and an exact realization of the programmed motion is impossible even when there are no other types of perturbations (constantly acting perturbations or parametric perturbations). Hence we have to remain content with just that the programmed motion can be made stable (asymptotically stable). And this is actually possible, since the solution of the problem of constructing the equations of motion according to a given programme of motion, which finally determines the control elements, is not unique. Using this ambiguity, we can complete the determination of the control elements in such a way that the given programmed motion is a stable motion of the system under consideration.

It follows from the above discussion that when the motion of a system is described by ordinary differential equations, the problem of the analytic construction of the systems of programmed motion must be represented in the following form.

Construct the equations of motion of a material system,

$$\ddot{y} = Y(y, \dot{y}, t), \tag{1.1}$$

where $y\,[y_1, \ldots, y_n]$ *is the vector of generalized coordinates of the system, so that the programmed motion with the given properties*

$$\Omega\colon\ \omega_\mu(y, \dot{y}, t) = 0 \quad (\mu = 1, \ldots, m \leqslant n) \tag{1.2}$$

is one of the possible motions of this system, and is stable with respect to these properties in the case of initial deviations from these properties.

In this case, it is assumed that the given properties (1.2) are compatible and independent, while the functions $\omega_\mu(y, \dot{y}, t)$ are bounded and differentiable in a certain given domain $G\{y, \dot{y}\}$ of the phase space for all $t \geqslant t_0$. The right-hand sides of the differential equations (1.1) are constructed in a class of functions which allow for the existence and uniqueness of the solution in a certain ε-neighbourhood $\Omega_\varepsilon \subset G\{y, \dot{y}\}$ of the programme.

The problem posed here is a quite general mathematical formulation of the problem of analytic construction of material systems of programmed motion for the case when the processes in the system are described by ordinary differential equations.

This problem is the initial one for programming the motion of a system with the help of additional controlling forces (open control system). As a matter of fact, the equations of motion of the system, corresponding to the given programme (1.2), are in this case constructed as follows:

$$\ddot{y} = Y_0(y, \dot{y}, t) + Y_u(y, \dot{y}, t), \tag{1.3}$$

where $Y_0(y, \dot{y}, t)$ is a known vector function defined by given generalized forces, $Y_u(y, \dot{y}, t)$ is the vector of the required controlling forces.

These equations can be used for a direct determination of controlling forces necessary for realizing a programmed motion.

This problem is also the initial problem for programming the motion of a system by changing its parameters in the course of motion (autonomous control system). In this case,

the structure of the equations of motion of the system under consideration is known, and has the form

$$\ddot{y} = Y_0 (y, \dot{y}, t, v), \tag{1.4}$$

where $v\ [v_1, \ldots, v_l]$ is the vector of the system parameters which are yet to be determined, and the corresponding equations of programmed motion of the system are constructed in the form (1.1). The required system parameters will be determined from the equations

$$Y_0 (y, \dot{y}, t, v) = Y (y, \dot{y}, t). \tag{1.5}$$

If the programming of the motion of a system is considered with the help of special control devices (closed control system) the initial problem in this case also is the problem of constructing the equations of motion. The equations are constructed in terms of the vector of the generalized output coordinates $u\ [u_1, \ldots, u_m]$ of the control devices in the form

$$\ddot{u} = U (y, \dot{y}, t, u, \dot{u}) \tag{1.6}$$

in accordance with the given programme of motion (1.1) of the material system itself. Here, it is assumed that the equations

$$\ddot{y} = Y_0 (y, \dot{y}, t, u, \dot{u}), \tag{1.7}$$

which describe the motion of the control object itself in conjunction with the control devices, are known.

It follows from the above discussion that the problem of the analytic construction of programmed motion systems is the initial general problem for realizingthe programmed motion by different methods.

It also follows from the above that the possible modifications of the problem under consideration on account of different programming methods are essentially extensions of some kind of formulation of the inverse problems of dynamics (the basic problem, the problems of restoration and closure of equations of motion) to the analytic construction of stable systems of programmed motion.

Sec. 2. Equations of Motion and the Stabiltiy of a Programme

The equations of programmed motion of material systems may be constructed, as shown above, directly on the basis of the equations of motion obtained while solving the corresponding inverse problems of dynamics (Secs. 4, 6, 8, Ch. 1). In this case, the programme Ω (1.2) itself, representing the given kinematic properties of the motion of a system, should be treated as an integral manifold of the required equations of motion.

Thus, in the cases where the motion of a system is programmed with the help of additional controlling forces and with the help of changes in the system parameters, the equations of motion are written as the solution of the basic problem of constructing the equations of motion and have the form (see Sec. 4, Ch. 1)

$$\ddot{y} = \frac{1}{\Gamma} \sum_{i, j=1}^{1, m} \Gamma_{ij} (\Phi_j - \varphi_j) \operatorname{grad}_{\dot{y}} \omega_i + Y^{\tau}. \qquad (2.1)$$

It should be recalled that the components of the vector function Y^{τ} in these equations are determined from the equalities

$$(\operatorname{grad}_{\dot{y}} \omega_{\mu} \cdot Y^{\tau}) = 0 \quad (\mu = 1, \ldots, m). \qquad (2.2)$$

In the general case, Eqs. (2.1) contain $n - m$ arbitrary functions $Y^{\tau}_{m+1} (y, \dot{y}, t), \ldots, Y^{\tau}_{n} (y, \dot{y}, t)$. The functions $\Phi_j = \Phi_j (\omega, y, \dot{y}, t)$ are also arbitrary, and satisfy only the conditions $\Phi_j (0, y, \dot{y}, t) = 0$ $(j = 1, \ldots, m)$.

It should be also recalled that

$$\Gamma = \det \|(\operatorname{grad}_{\dot{y}} \omega_i \cdot \operatorname{grad}_{\dot{y}} \omega_j)\|^m_m;$$

where Γ_{ij} is the cofactor of the (i, j)-th element of the determinant Γ;

$$\varphi_j = (\operatorname{grad}_{y} \omega_j \cdot \dot{y}) + \frac{\partial \omega_j}{\partial t}.$$

It should be observed that the equations of motion (2.1) formulated above can be used to determine the controlling forces, under the action of which the programmed motion

(1.2) becomes one of the possible motions of the material system under consideration. For this purpose, it is assumed that the right-hand side of Eq. (2.1) is the sum of the vector of generalized forces $Y_0(y, \dot{y}, t)$ and the vector of the required controlling forces Y_u. In this case, we get

$$Y_u = \frac{1}{\Gamma} \sum_{i,j}^{1,m} \Gamma_{ij} (\Phi_j - \varphi_j) \operatorname{grad}_{\dot{y}} \omega_i + Y^\tau - Y_0(y, \dot{y}, t). \quad (2.3)$$

Equations (2.1) can also be used to determine the change in the parameters of the system in the process of motion, so that the programmed motion (1.2) is one of the possible motions of the material system under consideration. For this purpose, it is assumed that the right-hand side of the equations of motion (2.1) is a known vector function $Y_0(y, \dot{y}, t, v)$, which depends on the vector v of the required system parameters. Then the solution of the equations

$$Y_0(y, \dot{y}, t, v) = \frac{1}{\Gamma} \sum_{i,j}^{1,m} \Gamma_{ij} (\Phi_j - \varphi_j) \operatorname{grad}_{\dot{y}} \omega_i + Y^\tau \quad (2.4)$$

defines the required parameters of the system.

It should be noted that in this case a question arises about the solvability of Eqs. (2.4) for the components of the vector v of the parameters. The requirements of the compatibility of Eqs. (2.4) and the existence of a solution (even if it is ambiguous) in the parameters of the system naturally impose additional conditions on the mathematical model of the system under consideration (on the structure of the vector function $Y_0(y, \dot{y}, t, v)$), as well as on the programme elements (i.e. on the functions $\omega_\mu(y, \dot{y}, t)$, $\mu = 1, \ldots, m$). It follows from this that while solving the problem of analytic construction of a system by programming with the help of parameters, we must first analyse the question of realizability of the motion of a system with given properties.

While programming a system with the help of control devices, the required equations of motion are written as the solutions of the problem of constructing the equations

closing the given equations and are of the form (see Sec. 8, Ch. 1)

$$\ddot{y} = Y_0(y, \dot{y}, t, u, \dot{u}),$$
$$\ddot{u} = \frac{1}{\Gamma^*} \sum_{i, j}^{1, m} \Gamma^*_{ij} (\Phi^*_j - \varphi^*_j) \operatorname{grad}_{\dot{u}} \omega_i + U^\tau, \tag{2.5}$$

where the first group of equations represents the equations of motion of the control object, while the second group describes the equations of motion of the control devices.

It should be recalled that here, the vector function U^τ is determined from the equations

$$(\operatorname{grad}_{\dot{u}} \omega_\mu \cdot U^\tau) = 0 \quad (\mu = 1, \ldots, m) \tag{2.6}$$

and contains in the general case $n - m$ arbitrary functions $U^\tau_{m+1}(y, \dot{y}, t), \ldots, U^\tau_n(y, \dot{y}, t)$; the functions $\Phi^*_j(\omega, \dot{\omega}, y, \dot{y}, t)$ are also arbitrary functions satisfying only the condition

$$\Phi^*_j(0, 0, y, \dot{y}, t) = 0 \quad (j = 1, \ldots, m).$$

The following notation has been used in Eqs. (2.5):

$$\Gamma^* = \det \|(\operatorname{grad}_{\dot{u}} \omega_i \cdot \operatorname{grad}_{\dot{u}} \omega_j)\|^m_m;$$

Γ^*_{ij} is the cofactor of the (i, j)-th element of the determinant Γ^*;

$$\varphi^*_j = (\operatorname{grad}_{\dot{y}} \dot{\omega}_j \cdot Y_0) + (\operatorname{grad}_y \dot{\omega}_j \cdot \dot{y}) + (\operatorname{grad}_u \dot{\omega}_j \cdot \dot{u}) + \frac{\partial \dot{\omega}_j}{\partial t}.$$

Equations (2.1) and (2.5) of the programmed motion systems contain, in the general case, n arbitrary functions Φ_μ, Y^τ_s and Φ^*_μ, U^τ_s $(\mu = 1, \ldots, m;\ s = m + 1, \ldots, n)$ respectively. Henceforth, we shall assume that these remaining arbitrary functions satisfy the conditions of the existence and uniqueness of the solution of the corresponding equations (2.1) and (2.5).

In this case, the motion of the material system under consideration will take place according to the given programme

(1.2), if only the representative point $M(y, \dot{y})$ at the initial instant of time t lies on the integral manifold Ω (1.2). In this case, $\Phi_\mu = 0$ in (2.1) and $\Phi_\mu^* = 0$ in (2.5) ($\mu = 1, \ldots, m$), while the remaining arbitrary functions may be determined with the help of additional conditions (for example, the optimality conditions) which are imposed on the motion of the representative point over the integral manifold.

Let us now assume that at the initial instant of time the representative point was outside the integral manifold Ω (1.2) in an infinitely small neighbourhood Ω_ε. Keeping this more real assumption in view, the remaining freedom in the choice of the above-mentioned arbitrary functions should be used for ensuring the stability of programmed motion of the material system under consideration.

For this purpose, we consider the programmed motion (1.2) of the system as unperturbed motion. Then all the possible motions of the system, starting from the state and with the velocities which are different from the programmed state and velocities, will be perturbed motions of the system. Let us compare all these motions with the unperturbed motions with respect to the values of the functions $\omega_\mu(y, \dot{y}, t)$ ($\mu = 1, \ldots, m$) defining the programme of the motion (1.2). In this case, the problem of stability of the programmed motion with respect to these functions will essentially be the problem of the stability of the integral manifold Ω (1.2), when the initial deviations of the functions ω_μ ($\mu = 1, \ldots, m$) are nonzero.

The corresponding equations of the perturbed motion for the case of programming by additional controlling forces and in the case of programming by varying the parameters of the system can be written in the form

$$\dot{\omega}_\mu = \Phi_\mu(\omega, y, \dot{y}, t) \quad (\mu = 1, \ldots, m). \qquad (2.7)$$

It should be noted that in view of the conditions imposed on the functions $\Phi_\mu(\omega, y, \dot{y}, t)$, the system of equations (2.7) has a trivial solution $\omega_1 = 0, \ldots, \omega_m = 0$, which is the integral manifold Ω (1.2) and corresponds to the unperturbed programmed motion of the material system under consideration. Hence, in order to obtain the required con-

ditions of stability (asymptotic stability), of the programmed motion with respect to the functions $\omega_\mu(y, \dot{y}, t)$, representing any characteristic indices of motion, it is sufficient to choose the functions $\Phi_\mu(\omega, y, \dot{y}, t)$ $(\mu = 1, \ldots, m)$ in such a way that the trivial solution $\omega_1 = 0, \ldots, \omega_m = 0$ of the system of equations (2.7) is stable (asymptotically stable).

In order to obtain the corresponding stability conditions, we can either use the method of characteristic numbers, or the method of Lyapunov's functions. In the general case, the method of Lyapunov's functions is usually applied when the equation of motion of a system is written by taking into account the deviation of the properties of motion from the programme, as well as by considering the state of the system itself (the case where $\Phi_\mu = \Phi_\mu(\omega, y, \dot{y}, t)$).

Suppose that we have constructed a certain function $V(\omega, y, \dot{y}, t)$ which is positive definite with respect to $\omega_1, \ldots, \omega_m$ in a certain domain

$$\sum_{\mu=1}^{m} \omega_\mu^2 \leqslant H, \quad G\{y, \dot{y}\} \times T\{t \geqslant t_0\}. \tag{2.8}$$

Let us find the time derivative $\dot{V}$ of this function, taking into account the equations of motion (2.1) and the equations of perturbed motion (2.7):

$$\dot{V} = \sum_{\mu=1}^{m} \frac{\partial V}{\partial \omega_\mu} \Phi_\mu + \sum_{\nu=1}^{n} \frac{\partial V}{\partial y_\nu} \dot{y}_\nu + \sum_{\nu=1}^{n} \frac{\partial V}{\partial \dot{y}_\nu} Y_\nu + \frac{\partial V}{\partial t}. \tag{2.9}$$

The conditions under which this derivative is constant negative with respect to $\omega_1, \ldots, \omega_m$ (or identically equal to zero) in the domain (2.8) will also be the required stability conditions for the programmed motion (1.2).

These conditions will serve as the initial conditions for the determination of the required functions $\Phi_1(\omega, y, \dot{y}, t), \ldots$ $\ldots, \Phi_m(\omega, y, \dot{y}, t)$ and $Y_{m+1}^{\tau}(y, \dot{y}, t), \ldots, Y_n^{\tau}(y, \dot{y}, t)$, appearing on the right-hand sides of the equations (2.1) constructed for the programmed motion of a material system.

It should be observed that Lyapunov's function V can be constructed, for example, as the following quadratic form:

$$V = \omega^t B(y, \dot{y}, t)\,\omega, \tag{2.10}$$

where ω^t is a vector function obtained by transposition of the matrix

$$\omega = \text{colon}\,[\omega_1, \ldots, \omega_m].$$

This form will be positive definite, if the $(n \times n)$ matrix $B(y, \dot{y}, t)$ with bounded and continuous elements satisfies Sylvester conditions in the domain $G\{y, \dot{y}\} \times T\{t \geqslant t_0\}$.

The method of obtaining the stability conditions is considerably simplified if the control is formed only according to the magnitude of the deviation of the properties of motion from the programmed properties (1.2), when the undefined functions Φ_μ are chosen in the form $\Phi_\mu' = \Phi_\mu(\omega, t)$ or in the form $\Phi_\mu = \Phi_\mu(\omega)$ $(\mu = 1, \ldots, m)$. The equations of the perturbed motion can then be written in the following forms respectively:

$$\dot{\omega}_\mu = \Phi_\mu(\omega, t)\ (\mu = 1, \ldots, m) \tag{2.11}$$

or

$$\dot{\omega}_\mu = \Phi_\mu(\omega) \quad (\mu = 1, \ldots, m). \tag{2.12}$$

It follows from this that the stability conditions in this case will be imposed only on the functions Φ_μ $(\mu = 1, \ldots$ $\ldots, m)$ and hence the stability will take place irrespective of the remaining undefined functions $Y^\tau_{m+1}, \ldots, Y^\tau_n$. Even the stability conditions themselves are much more simplified, since the absence of the generalized coordinates $y_1, \ldots, y_n$ and velocities $\dot{y}_1, \ldots, \dot{y}_n$ in the equations (2.11) and (2.12) of the perturbed motion allows us to directly use such methods of determining the stability conditions, as are quite well known in the theory of constructing stable systems (see Ch. 2).

Thus, for example, the method of characteristic numbers is quite useful in this case. Indeed, let us represent the corresponding equations of perturbed motion in the form

$$\dot{\omega}_s = \sum_{k=1}^{m} p_{sk}\omega_k + \Phi_s^{(2)} \quad (s = 1, \ldots, m), \tag{2.13}$$

where, for all $t \geqslant t_0$, $p_{sk} = p_{sk}(t)$ are bounded continuous coefficients of linear terms of expansion of the required functions $\Phi_s(\omega, t)$ $(s = 1, \ldots, m)$ in powers of $\omega_1, \ldots$ $\ldots, \omega_m$; $\Phi_s^{(2)} = \Phi_s^{(2)}(\omega, t)$ are the higher-order terms.

Then the required functions Φ_s may be constructed in such a way that the first-approximation equations of perturbed motion,

$$\dot{\omega}_s = \sum_{k=1}^{m} p_{sk}\omega_k \quad (s = 1, \ldots, m), \tag{2.14}$$

form a regular system, and the characteristic numbers of the solution of this system are positive. The programmed motion will then be asymptotically stable with respect to the given properties (1.2).

If, however, we assume that the coefficients of the linear terms of the functions Φ_s are constant, for the asymptotic stability of the programmed motion it is sufficient that all the roots of the characteristic equation

$$\det(-1)^m \| p_{sk} - \delta_{sk}\lambda \|_m^m = 0 \tag{2.15}$$

be situated on the left of the imaginary axis of the plane of the roots.

The above discussions about completing the equations of motion (2.1) in such a way that the existing programmed motion be stable, are also valid for the case when the system programming is carried out with the help of control devices. It should only be mentioned that in this case the undefined functions $\Phi_\mu^*(\omega, \dot{\omega}, y, \dot{y}, t)$ $(\mu = 1, \ldots, m)$ and $U_s^\tau(y, \dot{y}, t)$ $(s = m + 1, \ldots, n)$, appearing in the equations of motion (2.5), are determined in such a way that the trivial solution $\omega_1 = 0, \ldots, \omega_m = 0$ of the equations

$$\ddot{\omega}_\mu = \Phi_\mu^*(\omega, \dot{\omega}, y, \dot{y}, t) \quad (\mu = 1, \ldots, n) \tag{2.16}$$

of the perturbed motion are stable (asymptotically stable).

In conclusion of this section, it should be remarked that even in the general case, the conditions obtained for the stability of a programme usually allow us to determine only a part of the required arbitrary functions in the equations of motion. The remaining arbitrary functions are then determined from additional requirements [14, 15]. These require-

ments may be the condition that the motion of the corresponding representative point to or over the integral manifold be optimal in some sense, the conditions that a programme be realized with a certain given accuracy, various restrictions on the variation of the parameters and controlling forces, or the a priori imposed technical requirements while constructing the system of programmed motion itself.

Sec. 3. Programmed Motion of a Body with Varying Mass

As an example of construction of systems with programmed motion, let us consider the programmed motion of a body of varying mass [43] in the gravitational field.

3.1. Natural Equations of Motion of a Body with Varying Mass. Consider a heavy body with varying mass $m(t)$, whose centre of gravity O performs a programmed motion in the vertical plane:

$$\xi_0 = \xi_0(t), \quad \eta_0 = \eta_0(t). \tag{3.1.1}$$

Let us suppose that the centre of gravity of the body and the directions of the principal central axes of inertia remain unchanged relative to the initial geometrical configuration of the body during the variations in its mass. We choose the principal central axes of inertia as the coordinate axes x_1, y_1, z_1. Let $A = A(t)$, $B = B(t)$, $C = C(t)$ be the corresponding moments of inertia of the body. Henceforth, we shall assume that the body of varying mass has an axis of kinetic symmetry, which also is invariant relative to the initial geometrical configuration of the body. Along this axis we direct the z_1-axis ($A(t) = B(t)$). Let p, q and r be the projections of the instantaneous angular velocity $\boldsymbol{\omega}$ on the x_1-, y_1-, and z_1-axes.

We introduce another rectangular system of coordinates $Oxyz$, related to the trajectory of the centre of gravity of the body, where the z-axis is directed along the tangent to the trajectory, the y-axis along the principal normal, and the x-axis along the binormal. It should be noted that during the motion of the body, the trihedron $Oxyz$ rotates around the x-axis with an angular velocity $P = \dot{\tau}$, where τ is the angle between the tangent to the trajectory and the horizontal plane.

Suppose that the body moves in a certain resistive medium whose action on the body of varying mass is expressed by a dynamic pair: the principal vector **R** and the principal moment $\mathbf{M_c}$, calculated relative to the centre of gravity.

We lay off from the centre of gravity a unit vector **c** along the z_1-axis, and a unit vector **e** along the line of intersection of the plane passing through the z_1-axis and the instantaneous axis of rotation of the body, with the equatorial plane x_1Oy_1. We then have

$$\boldsymbol{\omega} = \omega_e \mathbf{e} + r\mathbf{c}, \tag{3.1.2}$$

while the moment of the momentum vector, calculated with respect to the centre of gravity of the body, is given by the expression

$$\mathbf{G} = A\omega_e \mathbf{e} + Cr\mathbf{c}, \tag{3.1.3}$$

or

$$\mathbf{G} = A\boldsymbol{\omega} + (C - A)\, r\mathbf{c}. \tag{3.1.4}$$

It should be noted that

$$\boldsymbol{\omega} = r\mathbf{c} - (\dot{\mathbf{c}} \times \mathbf{c}). \tag{3.1.5}$$

Hence, it follows from (3.1.3) that [10]

$$\mathbf{G} = Cr\mathbf{c} - A\,(\dot{\mathbf{c}} \times \mathbf{c}). \tag{3.1.6}$$

In accordance with the theorem on the moment of a body of varying mass, we get

$$\frac{d}{dt}(Cr\mathbf{c}) - \frac{d}{dt} A(\dot{\mathbf{c}} \times \mathbf{c}) = \mathbf{M_c} + \mathbf{M_r^{abs}} + \mathbf{M_1}, \tag{3.1.7}$$

where $\mathbf{M_r^{abs}}$ is the sum of the moments of reactive forces, caused by the absolute motion of the particles of varying mass, and $\mathbf{M_1}$ is the sum of moments of all inertia forces relative to the motion of these particles.

Proceeding from the principle of solidification [45], we can express Eq. (3.1.7) in the form

$$C\frac{d}{dt}(r\mathbf{c}) - A\,(\ddot{\mathbf{c}} \times \mathbf{c}) = \mathbf{M}c + \mathbf{M_r^{rel}} + \mathbf{M_1}, \tag{3.1.8}$$

where $\mathbf{M_r^{rel}}$ is the sum of the moments of the reactive forces, caused by the relative motion of the particles of varying mass.

Let x, y, z be the coordinates of the tip of the vector $\mathbf{c}$ in the system of coordinates, connected with the trajectory of the centre of gravity. Then projecting the equation (3.1.8) onto the axes of coordinates x, y, z, we get

$$\begin{aligned}
&C\,(rx)^{\cdot} - A\,(\ddot{y}z - y\ddot{z} - 2\dot{\tau}y\dot{y} - 2\dot{\tau}z\dot{z} - \ddot{\tau}y^2 - \ddot{\tau}z^2) \\
&\qquad\qquad = M_{\mathbf{c}x} + M_{\mathbf{r}x}^{\mathbf{rel}} + M_{1x}, \\
&C\,[(ry)^{\cdot} - \dot{\tau}rz] - A\,(\ddot{z}x - z\ddot{x} - 2\dot{\tau}x\dot{y} - \dot{\tau}^2xz + \ddot{\tau}xy) \\
&\qquad\qquad = M_{\mathbf{c}y} + M_{\mathbf{r}y}^{\mathbf{rel}} + M_{1y}, \\
&C\,[(rz)^{\cdot} + \dot{\tau}ry] - A\,(\ddot{x}y - x\ddot{y} + 2\dot{\tau}x\dot{z} + \dot{\tau}^2xy + \ddot{\tau}xz) \\
&\qquad\qquad = M_{\mathbf{c}z} + M_{\mathbf{r}z}^{\mathbf{rel}} + M_{1z},
\end{aligned} \tag{3.1.9}$$

where $(\quad)^{\cdot}$ indicates differentiation with respect to t.

Suppose that the resultant action of the resistive medium on the body is expressed by the force $\mathbf{R}\,(v, \eta)$ ($v = \sqrt{\dot{\xi}^2 + \dot{\eta}^2}$ is the velocity of the centre of gravity of the body), applied at the point D (centre of pressure) on the z_1-axis at the distance l from the centre of gravity O, and lying in the plane zOz_1, forming an angle δ with the z_1-axis, where the plane zOz_1 is inclined at an angle ψ to the vertical plane. In this case, the moment $\mathbf{M}$ of this force relative to the centre of gravity of the body will be directed at an angle ψ with respect to the x-axis and has a magnitude

$$M = Rl \sin \delta, \tag{3.1.10}$$

or

$$M = Rfl \sin \alpha,$$

where $f = \sin \delta / \sin \alpha$ and it is assumed that f depends only on the geometrical configuration of the body [46, 47], and α is the angle of inclination of the z_1-axis relative to the tangent to the trajectory (angle of attack).

It should be noted that

$$x = \sin \alpha \sin \psi, \quad y = -\sin \alpha \cos \psi, \quad z = \cos \alpha \tag{3.1.11}$$

and

$$M_x = -Rfly, \quad M_y = Rflx, \quad M_z = 0. \tag{3.1.12}$$

Let us examine the case when the sum of the moments of all forces at the centre of gravity of the body, caused by the variation of the mass of the body and the relative motion of the particles of varying mass, as well as the force of reaction of the medium on the body, are reduced to this very moment **M** in the course of the entire motion. In this case, the damping moment of the proper rotation of the body (around the z_1-axis) is balanced by the sum of the moments of the reactive forces and the inertial forces relative to the motion of the particles, and hence [48]

$$r = \text{const.}$$

In the case under consideration Eqs. (3.1.9) can be written in the form

$$\begin{aligned}
&Cr\dot{x} - A(y\ddot{z} - z\ddot{y} - 2\tau\dot{y}\dot{y} - 2\tau\dot{z}\dot{z} - \dot{\tau}y^2 - \dot{\tau}z^2) = -Rfly,\\
&Cr\dot{y} - Cr\tau z - A(z\ddot{x} - x\ddot{z} + 2\tau x\dot{y} - \tau^2 xz + \dot{\tau}xy) = Rflx,\\
&Cr\dot{z} + Cr\tau y - A(x\ddot{y} - y\ddot{x} + 2\tau x\dot{z} + \tau^2 xy + \dot{\tau}xz) = 0. \qquad (3.1.13)
\end{aligned}$$

Let us determine the position of the body relative to the coordinate system $Oxyz$ with the help of the longitude β and the latitude γ of the trace of the z_1-axis on a sphere of unit radius (the tip of the vector **c**), described around the centre of gravity of the body, where γ is the angle formed by the z_1-axis with its projection on the yOz plane, and β is the angle between the projection of the z_1-axis on the yOz plane and the z-axis.

In this case, the coordinates of the tip of the vector **c** are expressed in terms of these angles as follows:

$$x = \sin\gamma, \quad y = -\sin\beta\cos\gamma, \quad z = \cos\beta\cos\gamma. \qquad (3.1.14)$$

For such a choice of coordinates, the system of equations (3.1.9) can be reduced to the following system as a result of simple transformations:

$$\begin{aligned}
&A(\ddot{\beta} + \ddot{\tau})\cos\gamma - 2A(\dot{\beta} + \dot{\tau})\dot{\gamma}\sin\gamma + Cr\dot{\gamma} = Rfl\sin\beta, \qquad (3.1.15)\\
&A\ddot{\gamma} - Cr(\dot{\beta} + \dot{\tau})\cos\gamma + A(\dot{\beta} + \dot{\tau})^2\sin\gamma\cos\gamma = Rfl\cos\beta\sin\gamma.
\end{aligned}$$

The equations of rotational motion of an oblong projectile of a constant mass were presented in this very form in [46].

It should be observed that the system (3.1.15) of two equations obtained in this way completely replaces the system (3.1.13) of three equations, since one of the equations of the system (3.1.13) simply states that

$$x^2+y^2+z^2=1.$$

Henceforth, we shall assume that the programmed motion of a body of varying mass is accomplished by a corresponding time variation of the magnitude and direction of the resultant **T** of the reactive forces (pull), created by the change in the mass m ($\dot{m}(t)<0$). In this case, the magnitude and the direction of this force, necessary for the realization of a programmed motion, must satisfy the equations of motion of the centre of gravity of the body. The same equations also define the programmed value of the reactive force **R** of the medium, appearing on the right-hand sides of the equations (3.1.15) of the rotational motion of the body.

3.2. Stability of a Body on a Rectilinear Trajectory. Let us consider the case when the centre of gravity of a body of varying mass moves along a certain straight line and the quantity $a = Rfl$ remains constant in the course of the entire motion [28, 46].

In this case, the equations of the rotational motion of a body of varying mass have the form

$$\begin{gathered}A\ddot{\beta}\cos\gamma-2A\dot{\beta}\dot{\gamma}\sin\gamma+Cr\dot{\gamma}=a\sin\beta,\\ A\ddot{\gamma}+A\dot{\beta}^2\sin\gamma\cos\gamma-Cr\dot{\beta}\cos\gamma=a\cos\beta\sin\gamma.\end{gathered}\tag{3.2.1}$$

These equations have a trivial solution $\beta = 0$, $\gamma = 0$, and may be treated as the equations of perturbed motion of a body of varying mass relative to the angles β and γ.

This case for a body of constant mass (projectile) has been investigated in great details at present, and the stability condition has been obtained for the rotational motion of a projectile (Mayevskii's inequality [28, 46]).

In [28], this condition has been obtained with the help of Lyapunov's functions in the form of a bunch of integrals of the corresponding equations of perturbed motion. It is also possible to use this method for determining the sufficient conditions for the stability of the rotational motion of a body with varying mass.

For this purpose, we form the function

$$V=\frac{1}{2}ACr\dot{\beta}^2+2aA\dot{\beta}\gamma+\frac{1}{2}Car\gamma^2$$
$$+\frac{1}{2}ACr\dot{\gamma}^2-2aA\beta\dot{\gamma}+\frac{1}{2}Car\beta^2, \quad (3.2.2)$$

which is positive definite under the condition that

$$C^2r^2-4aA>\varepsilon>0 \quad (3.2.3)$$

and has an infinitely small upper limit (A and C are bounded, nonvanishing functions of time, and ε is a certain small number).

Differentiating the function V with respect to time, we obtain

$$\dot{V}=W_1+W_2+W_3, \quad (3.2.4)$$

where

$$W_1=ACr(\dot{\beta}\ddot{\beta}+\dot{\gamma}\ddot{\gamma})+2aA(\ddot{\beta}\gamma-\beta\ddot{\gamma})+Car(\beta\dot{\beta}+\gamma\dot{\gamma}),$$
$$W_2=\frac{1}{2}(AC)^{\boldsymbol{\cdot}}r\dot{\beta}^2+2a\dot{A}\dot{\beta}\gamma+\frac{1}{2}\dot{C}ar\gamma^2,$$
$$W_3=\frac{1}{2}(AC)^{\boldsymbol{\cdot}}r\dot{\gamma}^2-2a\dot{A}\beta\dot{\gamma}+\frac{1}{2}\dot{C}ar\beta^2.$$

Assuming that A and C are invariant, we obtain the following first integrals for Eq. (3.2.1):

$$\frac{1}{2}A(\dot{\gamma}^2+\dot{\beta}^2\cos^2\gamma)+a\cos\beta\cos\gamma=h,$$
$$A(\dot{\gamma}\sin\beta-\dot{\beta}\cos\beta\sin\gamma\cos\gamma)+Cr\cos\beta\cos\gamma=k, \quad (3.2.5)$$

which may be represented as follows:

$$\left.\begin{aligned}&\frac{1}{2}A(\dot{\beta}^2+\dot{\gamma}^2)+a\left(1-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)=V_1^{(4)}+h,\\&A(\beta\dot{\gamma}-\dot{\beta}\gamma)+Cr\left(1-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)=V_2^{(4)}+k,\end{aligned}\right\} \quad (3.2.6)$$

Here, $V_1^{(4)}$ and $V_2^{(4)}$ are holomorphic functions whose expansion in powers of β, γ, $\dot{\beta}$, and $\dot{\gamma}$ starts with terms of not lower than the fourth order.

Differentiating the expressions (3.2.6) with respect to time (assuming once again the invariance of A and B), we get

$$
\begin{aligned}
&A(\dot\beta\ddot\beta+\dot\gamma\ddot\gamma)-a(\beta\dot\beta+\gamma\dot\gamma)=W_1^{(3)},\\
&A(\beta\ddot\gamma-\ddot\beta\gamma)-Cr(\beta\dot\beta+\gamma\dot\gamma)=W_2^{(3)},
\end{aligned}
\tag{3.2.7}
$$

where $W_1^{(3)}$ and $W_2^{(3)}$ are holomorphic functions whose expansion in powers of β, γ, $\dot\beta$, and $\dot\gamma$ starts with terms of not lower than the third order.

It should be observed that the function W_1 in (3.2.4) is the derivative of V under the assumption that A and C remain unchanged. Hence, in view of the expressions (3.2.7), obtained from the first integrals (3.2.5) of Eqs. (3.2.1), and under the assumption that A and C remain constant, we get

$$W_1=CrW_1^{(3)}-2aW_2^{(3)}. \tag{3.2.8}$$

It follows hence that the sign of the time derivative of the function V is determined, in view of the equations (3.2.1) of the perturbed motion of a body of varying mass in a sufficiently small neighbourhood of the point $\beta=0$, $\gamma=0$, $\dot\beta=0$, and $\dot\gamma=0$, by the sign of the quadratic form W_2+ $+W_3$.

Consequently, if

$$(AC)^{\bullet}\,\dot{C}r^2-4a\dot{A}^2>\varepsilon>0,\quad \dot{C}<-\varepsilon, \tag{3.2.9}$$

the derivative of the function V is a negative definite function in the neighbourhood of this point.

Thus, the conditions

$$
\begin{gathered}
C^2r^2-4aA>\varepsilon>0,\\
(AC)^{\bullet}\dot{C}r^2-4a\dot{A}^2>\varepsilon,\\
\dot{C}<-\varepsilon
\end{gathered}
\tag{3.2.10}
$$

are the conditions for the stability of a rotating body of varying mass relative to the angles between the axis of kinetic symmetry of the body and the straight line which serves as the trajectory of its centre of gravity, as also relative to the rates of variation of these angles.

3.3. Stability of a Body on a Curvilinear Trajectory. Consider the case when the centre of gravity of a body of varying mass moves along a certain programmed curve (3.1.1) in the vertical plane. The equations of perturbed motion in this case form the following system:

$$\begin{aligned}
\dot{x}_1 &= p_{11}x_1 + p_{12}x_2 + p_{13}x_3 + p_{14}x_4 + X_1(x_3, x_4),\\
\dot{x}_2 &= p_{21}x_1 + p_{23}x_3 + p_{24}x_4 + X_2(x_3, x_4),\\
\dot{x}_3 &= x_1,\\
\dot{x}_4 &= x_2,
\end{aligned} \tag{3.3.1}$$

where x_1, x_2, x_3, and x_4 are respectively the perturbations of the rates of variation of the longitude β and the latitude γ of the trace of the z_1-axis on a sphere of unit radius, described about the centre of gravity of the body, and of their magnitudes ($x_1 = \dot{\beta} - \dot{\beta}_0$, $x_2 = \dot{\gamma} - \dot{\gamma}_0$, $x_3 = \beta - \beta_0$, and $x_4 = \gamma - \gamma_0$), $X_1(x_3, x_4)$ and $X_2(x_3, x_4)$ are holomorphic terms whose expansion in powers of x_3 and x_4 starts from terms of not lower than the first order,

$$\begin{aligned}
p_{11} &= 2\dot{\gamma}_0 \tan \gamma_0,\\
p_{12} &= 2(\dot{\beta}_0 + \dot{\tau}_0) \tan \gamma_0 - \frac{Cr}{A\cos\gamma_0},\\
p_{13} &= a_1 \frac{R_0 \cos \beta_0}{A \cos \gamma_0},\\
p_{14} &= 2(\dot{\beta}_0 + \dot{\tau}_0)\dot{\gamma}_0 + (\ddot{\beta}_0 + \ddot{\tau}_0) \tan \gamma_0,\\
p_{21} &= \left[\frac{Cr}{A} - 2(\dot{\beta}_0 + \dot{\tau}_0) \sin \gamma_0\right] \cos \gamma_0,\\
p_{23} &= a_1 \frac{R_0}{A} \sin \beta_0 \sin \gamma_0,\\
p_{24} &= a_1 \frac{R_0}{A} \cos \beta_0 \cos \gamma_0 - \frac{Cr}{A}(\dot{\beta}_0 + \dot{\tau}_0) \sin \gamma_0 - (\dot{\beta}_0 + \dot{\tau}_0)^2 \cos 2\gamma_0,\\
a_1 &= fl.
\end{aligned} \tag{3.3.2}$$

It should be noted that in these expressions the index "0" denotes the programme values of the variables, which are determined directly from the programme (3.1.1) itself (τ_0, R_0 (v_0, η_0)) and from the solution β_0, γ_0 of Eqs. (3.1.15), obtained by taking this programme into consideration. We

assume that the variables β_0 and γ_0 correspond to the unperturbed motion of a body of varying mass.

It is natural to assume that the programme of motion has been chosen in such a way that the angles β_0 and γ_0 remain sufficiently small in the course of the entire motion, and that the axis of kinetic symmetry of the body (the z_1-axis) moves with sufficiently small angular velocities $\dot{\beta}_0$ and $\dot{\gamma}_0$. Hence the sufficient conditions of stability (with respect to β, γ, $\dot{\beta}$, and $\dot{\gamma}$) of the rotary motion of a body, defined by the programme (3.3.1), are also sufficient for the stability of motion of a body on a curvilinear trajectory in the sense of the smallness of the angle between the axis of kinetic symmetry of the body and the tangent to the trajectory, as well as in the sense of the rate of variation of this angle.

The stability conditions for the unperturbed motion $\beta = \beta_0$, $\gamma = \gamma_0$, $\dot{\beta} = \dot{\beta}_0$, $\dot{\gamma} = \dot{\gamma}_0$ may be obtained, for example, with the help of the function

$$V = \frac{1}{2} ACrx_1^2 + 2Aa_1R_0x_1x_4 + \frac{1}{2} Ca_1R_0rx_4^2$$
$$+ \frac{1}{2} ACrx_2^2 - 2Aa_1R_0x_2x_3 + \frac{1}{2} Ca_1R_0rx_3^2, \qquad (3.3.3)$$

which is positive-definite under the condition that

$$C^2r^2 - 4Aa_1R > \varepsilon > 0 \qquad (3.3.4)$$

and has an infinitely small upper limit (A and C are bounded, nonvanishing functions of time, and ε is a certain small number).

On account of the equations (3.3.1) of perturbed motion, the sign of the derivative of this form for sufficiently small values of β_0, γ_0, $\dot{\beta}_0$, and $\dot{\gamma}_0$ is determined by the sign of the corresponding quadratic form. The condition of negative definiteness of this form can be represented in the form of the following inequalities in view of the smallness of β_0, γ_0, $\dot{\beta}_0$ and $\dot{\gamma}_0$:

$$\Delta_1 > \varepsilon > 0,$$
$$\Delta_1^2 > r^2\dot{\tau}_0^4 [(AC)^2 \Delta_1 + 4\Delta_2^2],$$
$$(AC) < -\varepsilon, \qquad (3.3.5)$$

where

$$\Delta_1 = r^2 (AC)^{\bullet} (Ca_1R_0)^{\bullet} - 4 [(Aa_1R_0)^{\bullet}]^2,$$
$$\Delta_2 = Aa_1R_0 (AC)^{\bullet} - AC (Aa_1R_0)^{\bullet}.$$

If these conditions are satisfied, the rotational motion of a body of varying mass will be stable with respect to the angles between the axis of kinetic symmetry of the body and the tangent to the trajectory of its centre of gravity, as well as with respect to the rates of variation of these angles, if only these angles and the rates of their variation are small in the programmed motion itself.

It should be also noted that the conditions (3.3.4) and (3.3.5) for $\dot{\tau}_0 = 0$ directly lead to the condition (3.2.10) for the stability of the motion of a body of varying mass on a rectilinear programmed trajectory.

Sec. 4. Construction of Functionals Which Can be Made Stationary in a Programmed Motion

In conclusion of this chapter on the construction of programmed motion systems, let us consider the problem of determining a set of integral functionals which assume a stationary value in the programmed motion of material systems.

It should be noted that the problems of finding functionals which assume extremal, or at least stationary, values on the solutions of equations of motion of mechanical systems were stated and solved in analytical mechanics as problems of establishing variational principles of dynamics, and in the calculus of variations as inverse problems in a given field of extremals [49-52]. In the theory of control of the motion of material systems, such problems were considered as the problems of determining the functionals which can be optimized by some choice of control functions [42, 53-55]. Moreover, it was assumed that the equations of motion of the system are known beforehand everywhere.

In the problem which we shall be considering here and later, the equations of motion of a system are not given beforehand, and must be constructed in accordance with the given law of motion or the given properties of motion from the appropriate conditions of realizability of the programmed motion. Thus, the problem of constructing stationary

functionals, which we shall be considering in this section, is also one of the possible versions of the inverse problems of dynamics.

4.1. Statement of the Problem. We consider a material system whose possible motions are described by differential equations of the type

$$\sum_{i=1}^{n} a_{\nu i}(q, \dot{q}, t)\, \ddot{q}_i = f_\nu (q, \dot{q}, t) \quad (\nu = 1, \ldots, n), \tag{4.1.1}$$

where $q\,[q_1, \ldots, q_n]$ is the vector of the system states (q_ν are the generalized coordinates), $a_{\nu i}$ and f_ν are bounded continuous functions differentiable with respect to all the variables q, $\dot{q}$, and t in a certain domain $G\{q, \dot{q}\}$ for all $t \geqslant t_0$ (f_ν are the generalized forces).

Suppose that the programme of motion of the system under consideration is given in the form of the vector of state of this system,

$$\varphi(t)\,[\varphi_1(t), \ldots, \varphi_n(t)], \tag{4.1.2}$$

composed of the laws of variation of generalized coordinates in the programmed motion, or in the form of a certain set of properties of programmed motion, described by the equalities

$$\Omega:\ \omega_\nu(q, \dot{q}, t) = 0 \quad (\nu = 1, \ldots, n), \tag{4.1.3}$$

where the vector function $\varphi(t)$ is twice differentiable for $t \geqslant t_0$, and the functions $\omega_\nu(q, \dot{q}, t)$ are independent and differentiable with respect to all arguments in the domain $G\{q, \dot{q}\}$ for $t \geqslant t_0$.

Consider the following problem [56, 57].

Find the generalized forces $f_\nu(q, \dot{q}, t)$ $(\nu = 1, \ldots, n)$, *under the action of which it is possible to realize the programmed motion of the system under consideration, given by the vector function* (4.1.2) *or by the equalities* (4.1.3), *and construct the functional*

$$J[q] = \int_{t_0}^{t} F(q, \dot{q}, t)\, dt, \tag{4.1.4}$$

which assumes a stationary value on the possible motions (including the programmed motion) of the system.

Here, it is assumed that the integrand $F(q, \dot{q}, t)$ is bounded and continuous, and all its partial derivatives with respect to all arguments in the domain $G\{q, \dot{q}\}$ exist up to the third order inclusive for $t \geqslant t_0$.

This problem is the initial problem for the analytic construction of material systems performing a given programmed motion. A solution of this problem will later help us to determine the parameters of the system itself and the control elements, as well as to establish beforehand certain characteristic indices of the possible motions of the system.

The solution of this problem can be reduced to the construction of differential equations (4.1.1) according to the given particular solution (4.1.2) or the given integral manifold (4.1.3) and to the construction of the corresponding functional (4.1.4) which assumes a stationary value for the solutions of the equations (4.1.1) constructed earlier. Since this type of inverse problems in the theory of differential equations and variational technique have an ambiguous solution, additional conditions must be imposed in order to determine eventually the right-hand sides of Eqs. (4.1.1) and the structure of the corresponding functional (4.1.4). One of such conditions may require that the given motion (the particular solution of (4.1.2) or the particular integrals (4.1.3)) be stable in the presence of the initial perturbations.

Thus, the problem formulated above is refined in the sense that the right-hand sides of the differential equations (4.1.1) and the structure of the functional (4.1.4) are determined by using the condition of stability of the programmed motion as well. These stability conditions, as well as additional conditions arising due to the requirement that the formulated problem should have a solution, will be established later while actually solving the problem.

It should be mentioned that in the general case, the programme of motion of the system is conditionally given by a set of certain indefinite functions $\varphi_\nu(t)$, $\omega_\nu(q, \dot{q}, t)$ $(\nu = 1, \ldots, n)$. While constructing the functional (4.1.4), as well as while establishing the stability conditions, some restrictions may be imposed on these functions as well. These

restrictions will be used for isolating the set of possible programmed motions of a material system.

4.2. Solution of the Problem According to a Given Law of Programmed Motion. Suppose that we are given the variations in the coordinates of a material system in the programmed motion

$$\Omega\colon\ q_\nu = \varphi_\nu(t) \quad (\nu = 1, \ldots, n), \tag{4.2.1}$$

where $\varphi_\nu(t)$ are twice differentiable functions for $t \geqslant t_0$.

The set of systems of differential equations, for which the given functions (4.2.1) are a particular solution, may be represented in the form [22]

$$\ddot{q}_\nu = \ddot{\varphi}_\nu(t) + \Phi_\nu(q, \dot{q}, t) \quad (\nu = 1, \ldots, n). \tag{4.2.2}$$

In these equations, the functions $\Phi_\nu(q, \dot{q}, t)$ represent the components of the generalized forces $f_\nu(q, \dot{q}, t)$ ($\nu = 1, \ldots, n$) under the action of which the given motion (4.2.1) of the system is possible. These functions satisfy the above-mentioned assumptions regarding the generalized forces, and the condition that

$$\Phi_\nu[\varphi(t), \dot{\varphi}(t), t] = 0, \quad (\nu = 1, \ldots, n).$$

For the rest, these functions are arbitrary.

Let us consider the restrictions on the determination of the structure of these functions which are still indefinite. These restrictions primarily arise while constructing the functional

$$J[q] = \int_{t_0}^{t} F[q, \dot{q}, \varphi(t), t]\, dt \tag{4.2.3}$$

which satisfies the above-mentioned assumptions regarding the integrand and assumes a stationary value for the solutions of the system of equations (4.2.2), including the one for the given programme of motion (4.2.1).

Indeed, the equations of the possible extremals corresponding to the functional (4.2.3) are of the form

$$\sum_{i=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\nu \partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\nu \partial q_i} \dot{q}_i + \frac{\partial^2 F}{\partial \dot{q}_\nu \partial t} = \frac{\partial F}{\partial q_\nu}$$
$$(\nu = 1, \ldots, n) \tag{4.2.4}$$

Comparing these equations with the equations of motion (4.2.2) constructed earlier, we get

$$\frac{\partial^2 F}{\partial \dot{q}_\nu \, \partial \dot{q}_i} = 0 \quad (\nu, i = 1, \ldots, n; \ \nu \neq i), \tag{4.2.5}$$

$$\frac{\partial^2 F}{\partial \dot{q}_\nu \, \partial q_i} = 0 \quad (\nu, i = 1, \ldots, n), \tag{4.2.6}$$

$$(\ddot{\varphi}_\nu (t) + \Phi_\nu) \frac{\partial^2 F}{\partial \dot{q}_\nu^2} = \frac{\partial F}{\partial q_\nu} - \frac{\partial^2 F}{\partial \dot{q}_\nu \, \partial t} \quad (\nu = 1, \ldots, n). \tag{4.2.7}$$

These equalities must be treated as equations for constructing the functional (4.2.3). From these equalities, it follows that the integrand of the required functional can be represented in the form of the sum

$$F = F_0 (q, t) + \sum_{j=1}^{n} F_j (\dot{q}_j, t), \tag{4.2.8}$$

where the terms $F_0 (q, t)$ and $F_j (\dot{q}_j, t)$ $(j = 1, \ldots, n)$ satisfy the conditions

$$(\ddot{\varphi}_\nu (t) + \Phi_\nu) \frac{\partial^2 F_\nu}{\partial \dot{q}_\nu^2} = \frac{\partial F_0}{\partial q_\nu} - \frac{\partial^2 F_\nu}{\partial \dot{q}_\nu \, \partial t} \quad (\nu = 1, \ldots, n). \tag{4.2.9}$$

Naturally, the conditions of existence and uniqueness of the solution of the equations of motion (4.2.2) as well as the requirement of the existence of the functions $F_0 (q, t)$ and $F_j (\dot{q}_j, t)$ $(j = 1, \ldots, n)$ which satisfy Eqs. (4.2.9) impose certain restrictions on the arbitrary functions $\Phi_\nu (q, \dot{q}, t)$ $(\nu = 1, \ldots, n)$.

The next set of restrictions imposed on these functions is defined while establishing the conditions of stability of programmed motion with respect to the generalized coordinates in the presence of initial perturbations of these coordinates.

In order to obtain these restrictions, we must first construct the appropriate equations of perturbed motion, which in the present case have the form

$$\ddot{x}_\nu = \Phi_\nu [\varphi (t) + x, \ \dot{\varphi} (t) + \dot{x}, \ t) \quad (\nu = 1, \ldots, n), \tag{4.2.10}$$

where $x\,[x_1, \ldots, x_n]$ is the vector of perturbations of generalized coordinates ($x_\nu = q_\nu - \varphi_\nu(t)$, $\nu = 1, \ldots, n$). After this, we use some criterion of stability [36] and determine the required conditions of stability of the trivial solution $x_1 = 0, \ldots, x_n = 0$ of the system of equations (4.2.10).

The conditions obtained in this way will be the restrictions for establishing the final structure of the right-hand sides of the constructed equations (4.2.2).

The following conclusions can be drawn from the above discussion.

The system of equations with given stable particular solutions (4.2.1) has the form (4.2.2), where the functions $\Phi_\nu(q, \dot{q}, t)$ are chosen in such a way that the trivial solution of Eqs. (4.2.10) is stable and the functional

$$J[q] = \int_{t_0}^{t} \left[F_0(q, t) + \sum_{\nu=1}^{n} F_\nu(\dot{q}_\nu, t) \right] dt, \qquad (4.2.11)$$

where $F_0(q, t)$ and $F_\nu(\dot{q}_\nu, t)$ satisfy the conditions (4.2.9), assumes a stationary value for the solutions of the system of equations (4.2.2) constructed earlier.

Let us consider one more particular case, where the result obtained can be used for specifying a more definite structure of the system of equations (4.2.2). We require that the functional

$$J[q] = \int_{t_0}^{t} \left[F_0(q, t) + \frac{1}{2} \sum_{\nu=1}^{n} \dot{q}_\nu^2 \right] dt \qquad (4.2.12)$$

assume a stationary value for the solutions of these equations. It should be noted that this case corresponds to the solution of Eqs. (4.2.5)-(4.2.7) for $\partial^2 F/\partial \dot{q}_\nu^2 = 1$ ($\nu = 1, \ldots, n$).

In the case under consideration, Eqs. (4.2.9) form the following system:

$$\frac{\partial F_0}{\partial q_\nu} = \Phi_\nu + \ddot{\varphi}_\nu(t) \quad (\nu = 1, \ldots, n), \qquad (4.2.13)$$

It follows from this that the functions Φ_ν may be represented in the form

$$\Phi_\nu = \frac{\partial U}{\partial q_\nu} \quad (\nu = 1, \ldots, n), \tag{4.2.14}$$

where U is a function of variables $q_1, \ldots, q_n$ and t.

Thus, in the case under consideration the system of equations (4.2.2) with the given particular solution (4.2.1) may be represented as follows:

$$\ddot{q}_\nu = \ddot{\varphi}_\nu(t) + \frac{\partial U(q, t)}{\partial q_\nu} \quad (\nu = 1, \ldots, n), \tag{4.2.15}$$

where the function $U(q, t)$ must be chosen in such a way that the conditions of the existence and uniqueness of the solutions of Eqs. (4.2.5) constructed earlier be satisfied, and that

$$\left.\frac{\partial U(q, t)}{\partial q_\nu}\right|_{q=\varphi(t)} = 0 \quad (\nu = 1, \ldots, n). \tag{4.2.16}$$

In this case, the functional

$$J[q] = \int_{t_0}^{t} \left[U(q, t) + \sum_{\nu=1}^{n} \ddot{\varphi}_\nu(t)\, q_\nu + \frac{1}{2} \sum_{\nu=1}^{n} \dot{q}_\nu^2 \right] dt \tag{4.2.17}$$

assumes a stationary value at the solutions of these equations.

It should be noted that the equations (4.2.15) constructed above are equations of motion of a material system under the action of forces having the force function

$$U'(q, t) + \sum_{\nu=1}^{n} \ddot{\varphi}_\nu(t)\, q_\nu,$$

while the functional (4.2.17) represents the action in Hamiltonian sense.

In order to define the conditions for the stability of the programmed motion (4.2.1), we construct the equations of the corresponding perturbed motion

$$\ddot{x}_\nu = \left.\frac{\partial U(q, t)}{\partial q_\nu}\right|_{q=\varphi(t)+x} \quad (\nu = 1, \ldots, n). \tag{4.2.18}$$

Further, we assume that the right-hand sides of these equations depend only on the perturbations $x_1, \ldots, x_n$ and are holomorphic functions in a certain domain

$$H\{\sum_{i=1}^{n} x_i^2 \leqslant H\}.$$

Moreover, the expansions of these functions in powers of perturbations start directly with terms of not lower than the first order in view of the assumptions (4.2.16) made above regarding U.

In this case, the equations (4.2.18) of perturbed motion form a system of equations of motion of a certain mechanical system in the neighbourhood of the equilibrium position $x_1 = 0, \ldots, x_n = 0$ under the action of potential forces. Consequently, the stability conditions imposed on the function U may be defined by Lagrange's theorem on the stability of the equilibrium position [28].

Thus, for example, if the function U is constructed in the form

$$U = \sum_{m_1+\ldots+m_n=m} a^{(m_1, \ldots, m_n)} (q_1 - \varphi_1(t))^{m_1} \ldots (q_n - \varphi_n(t))^{m_n} + \tilde{U}(q - \varphi(t)),$$

where m is a certain even number other than zero, and the summation is extended to all even numbers m_i not exceeding m, and $\tilde{U}(q - \varphi(t))$ are terms containing $q_\nu - \varphi_\nu(t)$ $(\nu = 1, \ldots, n)$ to powers higher than m, the negativeness of the constant coefficients $a^{m_1, \ldots, m_n}$ (some of these may vanish) is the condition for the stability of the corresponding programmed motion (4.2.1).

4.3. Motion of a Point of Varying Mass According to a Given Law. As an example, let us consider the solution of the following problem of a heavy point of varying mass [58].

Suppose that we are given the law of motion (programme of motion) of a point with varying mass $m(t)$ in the gravitational field along a parabola in the vertical plane

$$\Omega: \; y(t) = \dot{y}_0 t, \quad z(t) = -\frac{1}{2} g t^2 + \dot{z}_0 t, \tag{4.3.1}$$

where $y(t)$ and $z(t)$ are the range and altitude of the point in the programmed motion; $\dot{y}_0$ and $\dot{z}_0$ are the projections of the velocity $\mathbf{v}$ *of the point onto the y- and z-axes in the programmed motion for $t = 0$. Find the reactive force* **T** *under the action of which the point moves in a resistive medium according to the given law* (4.3.1), *and construct the functional which assumes a stationary value in this case.*

The equations of motion in the vertical plane of the point under consideration can be written by taking into account the force of resistance of the medium, $\mathbf{R}(v, z)$, and form the following system:

$$\begin{aligned} m\ddot{y} &= T\frac{\dot{y}}{v} - R(v, z)\frac{\dot{y}}{v}, \\ m\ddot{z} &= T\frac{\dot{z}}{v} - R(v, z)\frac{\dot{z}}{v} - mg. \end{aligned} \tag{4.3.2}$$

On the other hand, the system of equations having a given particular solution may be written in the following form:

$$\begin{aligned} \ddot{y} &= \Phi_1, \\ \ddot{z} &= \Phi_2 - g. \end{aligned} \tag{4.3.3}$$

It follows from this that the functions Φ_1 and Φ_2 in the problem under consideration must be defined as follows:

$$\begin{aligned} \Phi_1 &= \frac{1}{m}(T - R)\frac{\dot{y}}{v}, \\ \Phi_2 &= \frac{1}{m}(T - R)\frac{\dot{z}}{v}. \end{aligned} \tag{4.3.4}$$

Equations (4.3.3) constructed above have the given solution (4.3.1) if and only if the functions Φ_1 and Φ_2 vanish at this solution. Hence the given motion (4.3.1) is realized only if the following equality holds in the course of the entire motion:

$$T = R(v, z). \tag{4.3.5}$$

It should be noted that in this case the programmed motion (4.3.1) will be realized only in the case where the initial position and the initial velocity of the point coincide with

their programmed values. Otherwise the point will move along a parabola which will deviate from the programmed one (unstable programmed motion).

The functional (4.2.11), which can be made stable for the solutions of the system of equations (4.3.3) as well as for the given motion (4.3.1), has the following form in the case under consideration:

$$J[y, z] = \int_0^t [F_0(y, z, t) + F_1(\dot{y}, t) + F_2(\dot{z}, t)]\, dt, \qquad (4.3.6)$$

where the functions F_0, F_1, and F_2 satisfy the equations

$$\begin{aligned} \frac{\partial F_0}{\partial y} - \frac{\partial^2 F_1}{\partial \dot{y}\, \partial t} &= \frac{1}{m}(T - R)\frac{\dot{y}}{v}\frac{\partial^2 F_1}{\partial \dot{y}^2}, \\ \frac{\partial F_0}{\partial z} - \frac{\partial^2 F_2}{\partial \dot{z}\, \partial t} &= \left[\frac{1}{m}(T - R)\frac{\dot{z}}{v} - g\right]\frac{\partial^2 F_2}{\partial \dot{z}^2}. \end{aligned} \qquad (4.3.7)$$

If we assume that $F_0 = 0$, $F_2 = 0$, and $F_1 = \dot{y}^2/2$, these equalities are satisfied for the programmed motion (4.3.1).

Thus, the programmed motion (4.3.1) of a point of varying mass in the gravitational field is possible under the condition (4.3.5). In this case, the integral

$$J = \int_0^t \dot{y}^2\, dt,$$

defining the range of the point along the horizontal from its initial position to the position at the instant of time t has a stationary value.

4.4. Solution of the Problem According to the Given Properties of Programmed Motion. Suppose that we are given the properties of the programmed motion of a material system, described by the expressions

$$\Omega:\ \omega_\nu(q, \dot{q}, t) = 0 \qquad (\nu = 1, \ldots, m), \qquad (4.4.1)$$

where $\omega_\nu(q, \dot{q}, t)$ are independent functions which are bounded, continuous, and differentiable with respect to all the variables q, $\dot{q}$, and t in a certain interval $G\{q, \dot{q}\}$ for $t \geqslant t_0$.

The set of systems of differential equations, for which the given properties of motion (4.4.1) form a set of particular integrals, may be represented in the form [22]

$$\sum_{i=1}^{n} \frac{\partial \omega_\nu}{\partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^{n} \frac{\partial \omega_\nu}{\partial q_i} \dot{q}_i + \frac{\partial \omega_\nu}{\partial t} = \Phi_\nu (\omega, q, \dot{q}, t)$$

$$(\nu = 1, \dots, n) \qquad (4.4.2)$$

Here, the functions $\Phi_\nu (\omega, q, \dot{q}, t)$, which have not been defined so far, are the components of the generalized forces $f_\nu (q, \dot{q}, t)$ $(\nu = 1, \dots, n)$, under the action of which the motion with the given properties (4.4.1) is possible. In addition, these functions satisfy the assumptions made above regarding the generalized forces, as well as the condition that

$$\Phi_\nu (0, q, \dot{q}, t) = 0 \qquad (\nu = 1, \dots, n).$$

We require that the equations (4.4.2) constructed above be the equations of possible extremals of a certain functional

$$J[q] = \int_{t_0}^{t} F(q, \dot{q}, \omega, t)\, dt, \qquad (4.4.3)$$

which satisfies the assumptions made earlier regarding the required functionals.

Comparing Eqs. (4.4.2) with the corresponding Euler-Lagrange equations for the functional (4.4.3), we obtain*

$$\frac{\partial F}{\partial \dot{q}_\nu} = \omega_\nu (q, \dot{q}, t) \qquad (\nu = 1, \dots, n) \qquad (4.4.4)$$

and

$$\Phi_\nu = \frac{\partial F}{\partial q_\nu} \qquad (\nu = 1, \dots, n). \qquad (4.4.5)$$

* The expressions $\partial F/\partial q$ and $\partial F/\partial \dot{q}$ denote derivatives with respect to q and $\dot{q}$, which are contained in F both explicitly and in terms of q and $\dot{q}$ introduced into F by the given functions $\omega_\nu (q, \dot{q}, t)$.

Henceforth, we shall assume that the given integrals (4.4.1) satisfy the conditions

$$\frac{\partial \omega_\nu}{\partial \dot{q}_i} = \frac{\partial \omega_i}{\partial \dot{q}_\nu} \quad (i, \nu = 1, \ldots, n). \tag{4.4.6}$$

In this case, Eqs. (4.4.4) can be used to construct the function $F(q, \dot{q}, \omega, t)$ from the given particular integrals (4.4.1). In view of Eq. (4.4.5), this function also defines the right-hand sides of the required equations (4.4.2).

Thus, the system of equations with the given particular integrals (4.4.1) may be represented in the form

$$\sum_{i=1}^{n} \frac{\partial \omega_\nu}{\partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^{n} \frac{\partial \omega_\nu}{\partial q_i} \dot{q}_i + \frac{\partial \omega_\nu}{\partial t} = \frac{\partial F(q, \dot{q}, \omega, t)}{\partial q_\nu}, \tag{4.4.7}$$

where the function $F(q, \dot{q}, \omega, t)$ is constructed in such a way that the conditions of existence and uniqueness of the solution of the system (4.4.7) are satisfied, and that

$$\left. \frac{\partial F(q, \dot{q}, \omega, t)}{\partial q_\nu} \right|_{\omega=0} = 0 \quad (\nu = 1, \ldots, n). \tag{4.4.8}$$

Moreover, the functional (4.4.3) assumes a stationary value for the solutions of Eqs. (4.4.7).

In order to obtain the conditions of stability for the given properties of the programmed motion (4.4.1), we form an equation in perturbations of the functions ω_ν ($\nu = 1, \ldots, n$), assuming that these perturbations are caused by their initial deviations from zero (for $t = t_0$). In view of (4.4.7), these equations have the form

$$\dot{\omega}_\nu = \frac{\partial F(q, \dot{q}, \omega, t)}{\partial q_\nu} \quad (\nu = 1, \ldots, n) \tag{4.4.9}$$

and have a trivial solution $\omega_\nu = 0$ ($\nu = 1, \ldots, n$) in view of the condition (4.4.8).

Interpreting the given properties (4.4.1) of the programmed motion as mobile hypersurfaces in the phase space $\{q, \dot{q}\}$, we assume that these surfaces remain, for all $t \geqslant t_0$, in a certain bounded domain $G_1\{q, \dot{q}\} \subset G\{q, \dot{q}\}$.

Then, the requirement that the inequality

$$\sum_{\nu=1}^{n} \frac{\partial F(q, \dot{q}, \omega, t)}{\partial q_\nu} \omega_\nu (q, \dot{q}, t) \leqslant 0 \tag{4.4.10}$$

be satisfied in a certain domain $G_2 \{q, \dot{q}\} \subset G \{q, \dot{q}\}$, which is also bounded and includes $G_1 \{q, \dot{q}\}$ for all $t \geqslant t_0$, is the sufficient condition for the stability of the programmed motion in the sense that for an indefinitely small number ε, we can find such a number δ that for all initial values of the perturbations ω_ν satisfying the inequality

$$\sum_{\nu=1}^{n} \omega_\nu^2 (q_0, \dot{q}_0, i) \leqslant \delta,$$

the following inequality will be valid for all $t \geqslant t_0$:

$$\sum_{\nu=1}^{n} \omega_\nu^2 (q, \dot{q}, t) < \varepsilon.$$

If the properties of programmed motion are given by the expressions

$$\omega_\nu (q, t) = 0 \quad (\nu = 1, \ldots, n), \tag{4.4.11}$$

the set of required systems of differential equations may be represented in the form

$$\sum_{i=1}^{n} \frac{\partial \omega_\nu}{\partial q_i} \dot{q}_i + \frac{\partial \omega_\nu}{\partial t} = \Phi_\nu (\omega, q, t) \quad (\nu = 1, \ldots, n), \tag{4.4.12}$$

where $\Phi_\nu (\omega, q, t)$ are certain functions which have not been determined yet, and which satisfy the condition $\Phi_\nu (0, q, t) = 0$.

Requiring that Eqs. (4.4.12) be the equations of the possible extremals of the functional (4.4.3), we get

$$\frac{\partial^2 F}{\partial \dot{q}_\nu \partial \dot{q}_i} = 0, \quad \frac{\partial F}{\partial \dot{q}_\nu} = \omega_\nu (q, t), \quad \Phi_\nu = \frac{\partial F}{\partial q_\nu} \tag{4.4.13}$$

$$(i, \nu = 1, \ldots, n).$$

It follows hence that the functional (4.4.3) may be represented in the form

$$J[q]=\int_{t_0}^{t}\sum_{i=1}^{n}\dot{q}_i\omega_i(q,\ t)\,dt, \tag{4.4.14}$$

while the right-hand sides of Eqs. (4.4.12) are represented by the corresponding functions

$$\Phi_\nu=\sum_{i=1}^{n}\dot{q}_i\frac{\partial\omega_i(q,\ t)}{\partial q_\nu}\quad(\nu=1,\ \ldots,\ n), \tag{4.4.15}$$

which must vanish at given particular integrals (4.4.11) of these equations.

The stability condition for the programmed motion with respect to given properties (4.4.11) is that the inequality

$$\sum_{i,\ \nu}^{1,\ n}\dot{q}_i\frac{\partial\omega_i(q,\ t)}{\partial q_\nu}\omega_\nu(q,\ t)\leqslant 0 \tag{4.4.16}$$

be satisfied in a certain finite domain $G_2\subset G$ of the phase space $\{q,\ \dot{q}\}$ which contains the domain $G_1\ \{q,\ \dot{q}\}$ of the integrals (4.4.11) for all $t\geqslant t_0$.

4.5. Motion of a Point of Varying Mass with the Given Properties. By way of an example, let us consider the following problem [59].

Given the properties of motion of a heavy point of varying mass $m(t)$ in the vertical plane (programmed motion)

$$\begin{aligned}\Omega:\ \omega_1&\equiv\dot{y}-ay-\dot{y}_0=0,\\ \omega_2&\equiv\dot{z}-az-\dot{z}_0+gt=0,\end{aligned} \tag{4.5.1}$$

where y and z are the range and altitude of the point, $\dot{y}_0$ and $\dot{z}_0$ are the initial ($t=0$) projections of the velocity $\mathbf{v}$ *of the programmed motion onto the y and z axes, and a is a certain constant.*

Find the reactive force $\mathbf{T}$, *under the action of which the motion of a heavy point of varying mass with given properties (4.5.1) is one of its possible motions in a medium with resistance, and construct a functional which assumes a steady-state value in this case.*

In order to solve this problem, we first construct a system of equations of motion for which the given properties of motion (4.5.1) are particular solutions:

$$\ddot{y} = a\dot{y} + \Phi_1, \quad \ddot{z} = a\dot{z} - g + \Phi_2. \tag{4.5.2}$$

Comparing these equations with known equations (4.3.2) of motion of a point of varying mass in the gravitational field and taking into consideration the resistive force $R(v, z)$ of the medium, we get

$$\Phi_1 = \frac{1}{m}(T - R - amv)\frac{\dot{y}}{v}, \qquad \Phi_2 = \frac{1}{m}(T - R - amv)\frac{\dot{z}}{v}. \tag{4.5.3}$$

The condition that the functions Φ_1 and Φ_2 vanish for the programmed motion (4.5.1) determines the magnitude of the reactive force in the course of the motion:

$$T = R + amv. \tag{4.5.4}$$

The required functional (4.4.3) which assumes a stationary value in this case can be written in the form

$$J[y, z] = \int_0^t F(y, z, \dot{y}, \dot{z}, t)\,dt, \tag{4.5.5}$$

where the integrand satisfies the equalities

$$\begin{aligned}
\frac{\partial F}{\partial \dot{y}} &= \dot{y} - ay - \dot{y}_0, \\
\frac{\partial F}{\partial \dot{z}} &= \dot{z} - az - \dot{z}_0 + gt, \\
\frac{\partial F}{\partial y} &= \frac{1}{m}(T - R - amv)\frac{\dot{y}}{v}, \\
\frac{\partial F}{\partial z} &= \frac{1}{m}(T - R - amv)\frac{\dot{z}}{v}.
\end{aligned} \tag{4.5.6}$$

These equalities are satisfied for the programmed motion of the point if $F = \text{const}$. It follows from this that for the given motion (4.5.1) the necessary condition of the fastest transition of a point of varying mass in the gravitational

field from one point in the vertical plane to another is satisfied.

In conclusion of this section, let us mention that by making additional assumptions concerning the reactive force and the resistive force, we may draw conclusions about the stability of a programmed motion. Thus, for example, if these forces satisfy Eq. (4.5.4) for all the possible motions of a point of varying mass in the gravitational field in the vertical plane, the corresponding equations of perturbed motion (4.4.9) are written in the form

$$\dot{\omega}_1 = 0, \qquad \dot{\omega}_2 = 0, \tag{4.5.7}$$

and thus the motion is stable with respect to the programme (4.5.1) itself. In this case, it follows directly from the equations of motion (4.5.2) that if in addition $a < 0$, we get a stable programmed motion (4.5.1) also with respect to the coordinates of the point under consideration.

Chapter Four

INVERSE PROBLEMS OF DYNAMICS OF A RIGID BODY WITH ONE FIXED POINT IN GRAVITATIONAL FIELD

The inverse problems of dynamics for a rigid body with one fixed point were first considered by Chaplygin [8] and Goryachev [9]. But even the classic problem itself on the conditions for the existence of algebraic first integrals of the equations of motion of a rigid body with one fixed point in the gravitational field is a version of the inverse problems of dynamics of a rigid body, whose solution has been considered in a number of well-known works [61] and which with its several modifications remains even today the subject of extensive investigations.

In this chapter, we shall be considering the inverse problems of dynamics of a heavy rigid body with one fixed point. The property of motion, given in a general form, is treated as a fourth integral (or a particular integral) and the corresponding dynamic equations of motion are constructed over the entire integral manifold. Since the corresponding kinematic equations are known in this case, the problem of constructing the equations of motion in this case involves the closure of a system of kinematic equations by Euler's dynamic equations. After this, the constructed equations are used for formulating the necessary and sufficient conditions that are imposed on the mass geometry of the body in order that the equations of motion have a given solution. These conditions lead to the well-known classical cases of the existence of algebraic solutions and some particular cases of the solvability of the equations of motion of a heavy rigid body with one fixed point. In particular, the possibility of using these conditions for solving Chaplygin's problem on the existence of a linear integral with respect to the projections of the instantaneous angular velocity on the mobile coordinate axes. In this chapter, we shall also consider the possible developments of the problem under

investigation, towards the applications of the constructed dynamic equations for solving the problems of stability and control of the motion of a rigid body with one fixed point.

Sec. 1. Statement of the Problem

Consider the following inverse problem of dynamics of a rigid body with one fixed point [60].

Construct Euler's dynamic equations

$$\dot{x}_s = f_s(x_1, \ldots, x_6) \quad (s = 1, 2, 3) \tag{1.1}$$

from the following first integrals:

$$\Omega: \begin{cases} \omega_1 \equiv \frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_3^2) \\ \qquad + Mg(x_c x_4 + y_c x_5 + z_c x_6) = c_1, \\ \omega_2 \equiv Ax_1 x_4 + Bx_2 x_5 + Cx_3 x_6 = c_2, \\ \omega_3 \equiv \omega_3(x_1, \ldots, x_6) = c_3, \end{cases} \tag{1.2}$$

where x_1, x_2, and x_3 are the projections of the instantaneous angular velocity $\boldsymbol{\omega}$ of the body onto the principal axes x, y, z of the ellipsoid of inertia constructed at a fixed point O of the body, x_4, x_5, x_6 are the direction cosines of the mobile axes x, y, z with respect to the vertical, satisfying the kinematic equations

$$\begin{aligned} \dot{x}_4 &= x_3 x_5 - x_2 x_6 \equiv f_4, \\ \dot{x}_5 &= x_1 x_6 - x_3 x_4 \equiv f_5, \\ \dot{x}_6 &= x_2 x_4 - x_1 x_5 \equiv f_6; \end{aligned} \tag{1.3}$$

A, B, C are the moments of inertia of the body with respect to the principal axes x, y, z, Mg is the weight of the body, $\mathbf{r}_c$ $[x_c, y_c, z_c]$ is the radius vector of the centre of gravity of the body, $\omega_3(x_1, \ldots, x_6)$ is a function differentiable with respect to all the variables in a certain domain X $\{x_1, \ldots, x_6\}$ such that the first integrals of the system of equations (1.1) *and* (1.3)

$$\begin{gathered} \omega_0 \equiv x_4^2 + x_5^2 + x_6^2 = 1, \\ \omega_i(x_1, \ldots, x_6) = c_i \ (i = 1, 2, 3) \end{gathered} \tag{1.4}$$

are independent and compatible.

The problem formulated here is a problem of the closure of the incomplete system of differential equations (1.3) according to the given integral manifold Ω (1.2).

In order to solve this problem in accordance with the method described in Ch. 1, we formulate the necessary and sufficient conditions so that the given expressions (1.2) are the integrals of the system of equations (1.1) and (1.3), including the case when $c_3 = 0$. These conditions, obtained by differentiating the expression (1.2) with respect to time, and taking into account the differential equations (1.1) and (1.3), form a system of linear algebraic equations for the right-hand sides of the equations (1.3) being constructed. Solving these algebraic equations, we get the following required system of Euler's dynamic equations:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\Delta}[(\mathbf{G}\cdot\mathbf{f})(Cx_3\delta_2 - Bx_2\delta_3) - BC(\boldsymbol{\delta}\cdot\mathbf{f})f_4 \\ &\quad + Mg(\mathbf{r}_C\cdot\mathbf{f})(Bx_5\delta_3 - Cx_6\delta_2)] + \Phi_1, \\ \dot{x}_2 &= \frac{1}{\Delta}[(\mathbf{G}\cdot\mathbf{f})(Ax_1\delta_3 - Cx_3\delta_1) - CA(\boldsymbol{\delta}\cdot\mathbf{f})f_5 \\ &\quad + Mg(\mathbf{r}_C\cdot\mathbf{f})(Cx_6\delta_1 - Ax_4\delta_3)] + \Phi_2, \\ \dot{x}_3 &= \frac{1}{\Delta}[(\mathbf{G}\cdot\mathbf{f})(Bx_2\delta_1 - Ax_1\delta_2) - AB(\boldsymbol{\delta}\cdot\mathbf{f})f_6 \\ &\quad + Mg(\mathbf{r}_C\cdot\mathbf{f})(Ax_4\delta_2 - Bx_5\delta_1)] + \Phi_3, \end{aligned} \tag{1.5}$$

where $\mathbf{G}\,[Ax_1, Bx_2, Cx_3]$ is the kinetic moment of the body with respect to the fixed point O; $\Phi_1 = \Phi_2 = \Phi_3 = 0$ for the case when c_3 is an arbitrary constant ($c_3 \neq 0$); $\Phi_1 = BC\Phi f_4$, $\Phi_2 = CA\Phi f_5$, $\Phi_3 = AB\Phi f_6$ for the case when $c_3 = 0$; $\Phi = \Phi(x_1, \ldots, x_6)$ is an arbitrary function satisfying the condition $\Phi|_{\omega_3=0} = 0$; $\Delta = BC\delta_1 f_4 + CA\delta_2 f_5 + AB\delta_3 f_6 \neq 0$; $\mathbf{f}\,[f_4, f_5, f_6] = \boldsymbol{\zeta}_0 \times \boldsymbol{\omega}$; $\boldsymbol{\zeta}_0\,[x_4, x_5, x_6]$ is the unit vector along the vertical; $\boldsymbol{\delta}\,[\delta_4, \delta_5, \delta_6]$; $\delta_s = \partial\omega_3/\partial x_s$ ($s = 1, \ldots, 6$).

It should be noted that (1.5) directly leads to the well-known dynamic equations for Euler's case (if the integral $\omega_3 \equiv A^2x_1^2 + B^2x_2^2 + C^2x_3^2 = c_3$ is given), for Lagrange's case (if $\omega_3 \equiv x_3 = c_3$), as well as in Kovalevskaya's case $\left(\text{for } \omega_3 \equiv (x_1^2 - x_2^2 - nx_4)^2 + (2x_1x_2 - nx_5)^2 = c_3,\ n = \frac{Mg}{C}x_C\right)$.

Let us consider the variables $x_1, \ldots, x_6$ as the coordinates of a representative point M situated on the given integral manifold Ω (1.2), and require that the equations (1.5) constructed above be identically equal to the well-known Euler's dynamic equations for a heavy body with one fixed point. In this case, considering that $\Phi_1 = \Phi_2 = \Phi_3 = 0$ on the integral manifold Ω (1.2), we get

$$\begin{aligned}(B - C)\, x_2 x_3 + Mg\, (z_c x_5 - y_c x_6) &\equiv A\varphi_1, \\ (C - A)\, x_3 x_1 + Mg\, (x_c x_6 - x_c x_4) &\equiv B\varphi_2, \\ (A - B)\, x_1 x_2 + Mg\, (y_c x_4 - x_c x_5) &\equiv C\varphi_3,\end{aligned} \tag{1.6}$$

where $\varphi_1, \varphi_2, \varphi_3$ are the right-hand sides of Eqs. (1.5) without Φ_1, Φ_2, Φ_3.

The identities (1.6) represent the necessary and sufficient conditions for the existence of first integrals of the system of dynamic equations (1.1) and the kinematic equations (1.3), representing the motion of a body with one fixed point in the form (1.2) and may serve as the initial equations for solving the following problem.

Find the conditions imposed on the mass geometry (A, B, C, x_c, y_c, z_c) *of a body and the dynamic indices of motion of the body* (*holonomic and non-holonomic constraints, servoconstraints*), *which must be satisfied in order that the given first integrals* (1.2) *exist for the system of equations* (1.1) *and* (1.3).

It should be noted that the identities (1.6) may be used for solving other problems as well, for example, to determine the necessary constraints for the given first integrals and given restrictions concerning the mass geometry of the body, as well as to determine the particular solutions for given constraints applied to the body and for a given mass distribution of a heavy rigid body.

Sec. 2. Existence of Classical Integrals

We shall show how the identities (1.6) can be used for obtaining the conditions under which there exists some classical fourth integral for the equations of motion of a heavy rigid body with one fixed point.

Let us assume that the following integral is given for the system of equations (1.1) and (1.3):

$$\omega_3 \equiv \frac{1}{2}(A^2x_1^2 + B^2x_2^2 + C^2x_3^2) = c_3. \tag{2.1}$$

We then have

$$\delta_1 = A^2x_1,\ \delta_2 = B^2x_2,\ \delta_3 = C^2x_3,$$
$$\delta_4 = \delta_5 = \delta_6 = 0,\ \Delta = ABC\,(\mathbf{G}\cdot\mathbf{f}).$$

Euler's dynamic equations (1.5) in this case assume the form

$$\begin{aligned} A\dot{x}_1 &= (B - C)\,x_2x_3 + Mg\,(Cx_3x_5 - Bx_2x_6)\,(\mathbf{r}_c\cdot\mathbf{f})/(\mathbf{G}\cdot\mathbf{f}),\\ B\dot{x}_2 &= (C - A)\,x_3x_1 + Mg\,(Ax_1x_6 - Cx_3x_4)\,(\mathbf{r}_c\cdot\mathbf{f})/(\mathbf{G}\cdot\mathbf{f}),\\ C\dot{x}_3 &= (A - B)\,x_1x_2 + Mg\,(Bx_2x_4 - Ax_1x_5)\,(\mathbf{r}_c\cdot\mathbf{f})/(\mathbf{G}\cdot\mathbf{f}), \end{aligned} \tag{2.2}$$

while the identities (1.6), which are the necessary and sufficient conditions for the existence of the integral (2.1) of the system of equations (1.3) and (2.2), have the form

$$\begin{aligned} (z_cx_5 - y_cx_6)\,(\mathbf{G}\cdot\mathbf{f}) &\equiv (Cx_3x_5 - Bx_2x_6)\,(\mathbf{r}_c\cdot\mathbf{f}),\\ (x_cx_6 - z_cx_4)\,(\mathbf{G}\cdot\mathbf{f}) &\equiv (Ax_1x_6 - Cx_3x_4)\,(\mathbf{r}_c\cdot\mathbf{f}),\\ (y_cx_4 - x_cx_5)\,(\mathbf{G}\cdot\mathbf{f}) &\equiv (Bx_2x_4 - Ax_1x_5)\,(\mathbf{r}_c\cdot\mathbf{f}). \end{aligned} \tag{2.3}$$

It follows from the above identities that the mass distribution of a body corresponding to Euler's case, in which the centre of gravity of the body is situated at a fixed point $x_c = y_c = z_c = 0$, is the sufficient condition for the existence of the given integral (2.1).

It can be shown by following the same line of argument that the sufficient condition for the existence of the integral

$$\omega_3 \equiv x_3 = c_3$$

is a mass distribution under which

$$A = B,\ x_c = y_c = 0,\ z_c \neq 0 \quad \text{(Lagrange's case).}$$

A mass distribution under which

$$A = B = 2C,\ y_c = z_c = 0,\ x_c \neq 0 \quad \text{(Kovalevskaya's case)}$$

is the sufficient condition for the existence of the integral

$$\omega_3 \equiv (x_1^2 - x_2^2 - nx_4)^2 + (2x_1x_2 - nx_5)^2 = c_3 \quad \left(n = \frac{Mg}{C}\,x_c\right).$$

It should be noted that the identities (1.6) can also be used to solve such problems, in which the mass distribution is determined in accordance with given integrals when a given constraint exists.

Let us suppose, for example, that the fourth integral of the equations of motion of a heavy body with one fixed point is given in the following form:

$$\omega_3 \equiv x_3 (x_1^2 + x_2^2) - n x_1 x_6 = c_3 \quad \left(n = \frac{Mg}{C} x_c\right) \tag{2.4}$$

while the following condition is satisfied in the course of the motion:

$$4 (x_1 x_4 + x_2 x_5) + x_3 x_6 = 0. \tag{2.5}$$

In view of the existing integral $\omega_2 = c_2$, this condition may be satisfied if $A = B = 4C$. This means that the kinetic moment of the body with respect to a fixed point remains in a horizontal plane $(\mathbf{G} \cdot \boldsymbol{\zeta}_0) = 0$ in the course of the motion.

In the present case

$$\delta_1 = 2x_1 x_3 - n x_6, \quad \delta_2 = 2x_2 x_3,$$
$$\delta_3 = x_1^2 + x_2^2, \; \delta_4 = \delta_5 = 0, \; \delta_6 = -n x_1,$$
$$\Delta = 8C^2 [2 (x_1^2 + x_2^2) - x_3^2] f_6 - 4nC^2 x_6 f_4 \neq 0,$$
$$(\mathbf{G} \cdot \mathbf{f}) = -3C x_3 f_6.$$

Substituting these expressions into the identities (1.6), we get

$$8C^2 (y_c f_5 + z_c f_6) [2x_5 (x_1^2 + x_2^2) - x_2 x_3 x_6] = \Delta (z_c x_5 - y_c x_6),$$
$$8C^2 (y_c f_5 + z_c f_6) [x_1 x_3 x_6 - 2x_4 (x_1^2 + x_2^2)] = -\Delta z_c x_4,$$
$$8C^2 (y_c f_5 + z_c f_6) x_3 f_6 = \Delta y_c x_4. \tag{2.6}$$

This directly leads to the corresponding position of the centre of mass of the body,

$$y_c = z_c = 0,$$

for which the given integral (2.4) exists if we take into account the given constraint (2.5).

It should be observed that the mass distribution of the body ($A = B = 4C$, $y_c = z_c = 0$, $x_c \neq 0$) corresponds to the special case of the integrability of the equations of motion of a heavy rigid body with one fixed point, known as the Goryachev-Chaplygin case [61].

Sec. 3. Existence of Linear Integrals

An investigation of the integrals of the equations of motion of a rigid body with one fixed point, which are linear with respect to the projections of the instantaneous angular velocity, has been carried out by Goryachev [9] for the general case when the body is acted upon by arbitrary forces and there are no constraints on the mass distribution. He assumes that the coefficients of linear integrals are functions of Euler's angles. However, it is well known that under certain assumptions concerning the mass distribution of a body and the moments of forces acting on the body, the equations of motion of a body with one fixed point have a linear integral with constant coefficients. Thus, for example, the integral corresponding to the equations of motion of a heavy rigid body for $A = B$, $x_c = y_c = 0$, $z_c \neq 0$ (Lagrange's case) is given by $\omega_3 \equiv x_3 = c_3$. The problem of investigating the cases for which linear partial integrals with constant coefficients exist for the equations of motion of a heavy rigid body with one fixed point was first analyzed by Chaplygin [8].

We shall consider some cases of solving this problem by first constructing the linear integral itself, and then using the method described above for constructing the equations of motion according to the given integrals.

Suppose that the integral of the equations of motion of a heavy rigid body with one fixed point is given by

$$\omega_3 \equiv Aax_1 + Bbx_2 + Ccx_3 = c_3, \tag{3.1}$$

where a, b, and c are constants which are yet to be determined.

In the present case, we have

$$\delta_1 = Aa,\ \delta_2 = Bb,\ \delta_3 = Cc,\ \delta_4 = \delta_5 = \delta_6 = 0,$$
$$\Delta = ABC\,(af_4 + bf_5 + cf_6) \neq 0.$$

In order to solve this problem, we shall first construct the dynamic equations (1.5) having known integrals $\omega_1 = c_1$, $\omega_2 = c_2$ and the given linear integral (3.1).

These equations form the following system:

$$\dot{x}_1 = \frac{BC}{\Delta}\left[(\mathbf{G}\cdot\mathbf{f})(bx_3 - cx_2) + Mg(\mathbf{r}_c\cdot\mathbf{f})(cx_5 - bx_6)\right],$$
$$\dot{x}_2 = \frac{CA}{\Delta}\left[(\mathbf{G}\cdot\mathbf{f})(cx_1 - ax_3) + Mg(\mathbf{r}_c\cdot\mathbf{f})(ax_6 - cx_4)\right], \quad (3.2)$$
$$\dot{x}_3 = \frac{AB}{\Delta}\left[(\mathbf{G}\cdot\mathbf{f})(ax_2 - bx_1) + Mg(\mathbf{r}_c\cdot\mathbf{f})(bx_4 - ax_5)\right].$$

Next, we form the necessary and sufficient conditions for the existence of the given linear integral (3.1). These conditions, obtained by equating the right-hand sides of Eqs. (3.2) to the corresponding right-hand sides of the known Euler's equations, lead to the following systems of identities:

$$(\mathbf{r}_c\cdot\mathbf{f})(cx_5 - bx_6) \equiv (z_c x_5 - y_c x_6)(af_4 + bf_5 + cf_6),$$
$$(\mathbf{r}_c\cdot\mathbf{f})(ax_6 - cx_4) \equiv (x_c x_6 - z_c x_4)(af_4 + bf_5 + cf_6), \quad (3.3)$$
$$(\mathbf{r}_c\cdot\mathbf{f})(bx_4 - ax_5) \equiv (y_c x_4 - x_c x_5)(af_4 + bf_5 + cf_6)$$

and

$$(\mathbf{G}\cdot\mathbf{f})(bx_3 - cx_2) \equiv (B - C)\, x_2 x_3 \frac{\Delta}{ABC},$$
$$(\mathbf{G}\cdot\mathbf{f})(cx_1 - ax_3) \equiv (C - A)\, x_3 x_1 \frac{\Delta}{ABC}, \quad (3.4)$$
$$(\mathbf{G}\cdot\mathbf{f})(ax_2 - bx_1) \equiv (A - B)\, x_1 x_2 \frac{\Delta}{ABC}.$$

The system of identities (3.3) can be used to determine the required coefficients of the integral

$$a = x_c, \quad b = y_c, \quad c = z_c.$$

In this case, the identities (3.4), which are the necessary and sufficient conditions for the existence of the linear integral (3.1), can be written in the following form:

$$(\mathbf{G}\cdot\mathbf{f})(y_c x_3 - z_c x_2) \equiv (B - C)\, x_2 x_3 (\mathbf{r}_c\cdot\mathbf{f}),$$
$$(\mathbf{G}\cdot\mathbf{f})(z_c x_1 - x_c x_3) \equiv (C - A)\, x_3 x_1 (\mathbf{r}_c\cdot\mathbf{f}), \quad (3.5)$$
$$(\mathbf{G}\cdot\mathbf{f})(x_c x_2 - y_c x_1) \equiv (A - B)\, x_1 x_2 (\mathbf{r}_c\cdot\mathbf{f}).$$

The identities (3.5) hold when one of the following conditions is satisfied:

1. $A = B, \quad x_c = y_c = 0, \quad z_c \neq 0;$ (3.6)
2. $2A = B, \quad x_c = z_c = 0, \quad y_c \neq 0;$ (3.7)
3. $A(B - C)x_c^2 - C(A - B)z_c^2 = 0, \quad y_c = 0;$ (3.8)
4. $A = B = C.$ (3.9)

These conditions lead to the well-known integrability criteria by Lagrange ($\omega_3 = c_3$), Steklov-Bobylev ($\omega_3 = c_3$), Hess-Appelrot ($\omega_3 = 0$) and to the case of complete symmetry of a rigid body ($\omega_3 = c_3$) respectively [61].

It should be noted that the identities (3.5) can also be considered as equations of constraints for which the linear integral

$$\omega_3 \equiv Ax_c x_1 + By_c x_2 + Cz_c x_3 = c_3 \tag{3.10}$$

holds. Let us represent these identities in the form

$$[(B - C)x_c x_2 x_3 + (C - A)y_c x_3 x_1 + (A - B)z_c x_1 x_2] f_i = 0 \quad (i = 4, 5, 6). \tag{3.11}$$

Hence it follows that for the equations of motion of a heavy body with one fixed point there exists the linear integral (3.10) if the kinematic elements of motion are subjected to constraints described by the equations

$$f_i = 0 \quad (i = 4, 5, 6) \tag{3.12}$$

or the equation

$$(B - C)x_c x_2 x_3 + (C - A)y_c x_3 x_1 + (A - B)z_c x_1 x_2 = 0. \tag{3.13}$$

The constraints (3.12) describe permanent rotation of a body about the vertical, while the constraint (3.13) corresponds to the case of permanent rotations established by Mlodzeevskii and Staude [61].

Sec. 4. Controlled Motion of a Rigid Body

The dynamic equations (1.5), constructed from the known integrals $\omega_1 = c_1$, $\omega_2 = c_2$ and from the conditionally given integral $\omega_3 = c_3$ may also be used for solving the pro-

blems of controlling the motion of a heavy rigid body with one fixed point.

Consider the following problem.

The law of motion of a heavy rigid body with one fixed point is given by

$$\omega_3 (x_1, \ldots, x_6) = 0. \tag{4.1}$$

Find the controlling moment $\mathscr{L}_u$ *such that the motion with the given property* (4.1) *is one of the possible motions of the body under investigation.*

In this problem, the motion of the body with the given property (4.1) will be programmed motion, while the property itself will be the programme of motion.

It should be noted that the programme (4.1) must be one of the particular solutions of the equations (1.1) and (1.3) of the motion of the body. Let us assume that this integral (4.1) is not a consequence of the first integrals $\omega_0 = 1$, $\omega_1 = c_1$, $\omega_2 = c_2$ appearing in this case, and that the function $\omega_3 (x_1, \ldots, x_6)$ itself is differentiable with respect to all the variables in a certain domain $X \{x_1, \ldots, x_6\}$ including all the points in the ε-neighbourhood of the surface (4.1).

In order to solve this problem, let us construct the appropriate dynamic equations (1.1) in the following form:

$$\begin{aligned} A\dot{x}_1 &= (B - C)\, x_2 x_3 + Mg\,(z_c x_5 - y_c x_6) + \mathscr{L}_{ux}, \\ B\dot{x}_2 &= (C - A)\, x_3 x_1 + Mg\,(x_c x_6 - z_c x_4) + \mathscr{L}_{uy}, \\ C\dot{x}_3 &= (A - B)\, x_1 x_2 + Mg\,(y_c x_4 - x_c x_5) + \mathscr{L}_{yz}, \end{aligned} \tag{4.2}$$

where $\mathscr{L}_{ux}$, $\mathscr{L}_{uy}$, $\mathscr{L}_{uz}$ are the projections of the required controlling moment onto the mobile axes.

In this case, the construction of equations (4.2) in accordance with the given integrals directly solves the problem formulated above.

As a matter of fact, by formulating the necessary and sufficient conditions for the expressions $\omega_1 = c_1$, $\omega_2 = c_2$, and $\omega_3 = 0$ to be the integrals of the system of equations (4.2), and by solving them for the required projections of the

controlling moment, we directly obtain

$$\begin{aligned}
\mathcal{L}_{ux} &= -\frac{ABC}{\Delta} N f_4 + \Phi_1, \\
\mathcal{L}_{uy} &= -\frac{ABC}{\Delta} N f_5 + \Phi_2, \\
\mathcal{L}_{uz} &= -\frac{ABC}{\Delta} N f_6 + \Phi_3,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
N = {} & [(B-C)\, x_2 x_3 + Mg\, (z_c x_5 - y_c x_6)] \frac{\delta_1}{A} \\
& + [(C-A)\, x_3 x_1 + Mg\, (x_c x_6 - z_c x_4)] \frac{\delta_2}{B} \\
& + [(A-B)\, x_1 x_2 + Mg\, (y_c x_4 - x_c x_5)] \frac{\delta_3}{C} \\
& + \delta_4 f_4 + \delta_5 f_5 + \delta_6 f_6 ; \\
& \Phi_1 = f_4 \Phi, \quad \Phi_2 = f_5 \Phi, \quad \Phi_3 = f_6 \Phi ;
\end{aligned}$$

$\Phi = \Phi\, (x_1, \ldots, x_6)$ is an arbitrary function which vanishes on the surface (4.1).

It should be observed that if the given particular solution (4.1) is a corollary of four independent first integrals of motion of a heavy body with one fixed point in the known cases of integrability (both classical and particular), and if corresponding conditions for the mass geometry of the body are satisfied, then

$$\mathcal{L}_{ux} = f_4 \Phi, \quad \mathcal{L}_{uy} = f_5 \Phi, \quad \mathcal{L}_{uz} = f_6 \Phi. \tag{4.4}$$

From this it follows that the controlling moment vanishes in this case if the corresponding representative point $M\, (x_1, \ldots, x_6)$ lies on the surface (4.1).

As an example of the above problem of controlling the motion of a heavy rigid body with one fixed point, let us consider the problem of imparting the generalized precession to a body. This problem is formulated as the following inverse problem of dynamics of a rigid body.

Find the moment of controlling forces, under the action of which a heavy rigid body with one fixed point has a generalized precession in the sense that in the course of motion, the vector of instantaneous angular velocity $\boldsymbol{\omega}$ *remains in the plane passing*

through the unit vector ζ_0 along the vertical and the radius vector $\mathbf{r}_c$ of the centre of gravity.

It should be observed that the precession of the body in this sense may take place even without additional controlling forces, but with additional constraints only, which are directly imposed on the mass geometry and on the angular velocity of the body [62].

By applying the method of constructing the equations from the given properties of motion, we can determine the moment of controlling forces under the action of which the generalized precession of the body can be realized without imposing any additional constraints on the mass geometry and on the kinematic indices of motion.

The particular solution, which is given in the present case and which describes the generalized precession, can be written in the following form:

$$\omega_3 \equiv (\mathbf{r}_c \omega \zeta_0) = 0. \tag{4.5}$$

From this it follows that

$$\delta_1 = z_c x_5 - y_c x_6, \quad \delta_2 = x_c x_6 - z_c x_4,$$
$$\delta_3 = y_c x_4 - x_c x_5, \quad \delta_4 = y_c x_3 - z_c x_2, \quad \delta_5 = z_c x_1 - x_c x_3,$$
$$\delta_6 = x_c x_2 - y_c x_1.$$

Substituting these values into (4.3), we can find the projections $\mathcal{L}_{ux}$, $\mathcal{L}_{uy}$, $\mathcal{L}_{uz}$ of the moment under the action of which the generalized precession of a heavy rigid body can be realized. The generalized precession under the action of this moment will take place if $\omega_3 (x_1, \ldots, x_6)|_{t=t_0} = 0$ at the initial instant of time.

Sec. 5. On the Stability and Further Development of the Problem

The problem of determining the controlling moment under the action of which the programmed motion of a heavy rigid body with one fixed point is one of its possible motions, is usually formulated in conjunction with the problem of stability of the programmed motion with respect to some indices of motion. This is due to the fact that the controlling moment, defined by the expressions (4.3), is just necessary for realizing a given programme. The given programmed

motion in this case will be realized only if the representative point $M(x_1, \ldots, x_6)$ is situated on the surface (4.1) at the initial instant of time. In real motion, it is natural to assume that there will always be some initial deviations from the given programme. That is why the problem of choosing the controlling moment is formulated in such a way that the given programmed motion is stable (asymptotically stable) in the presence of the initial perturbations relative to the indices of motion given a priori, for example, relative to the coordinates and velocities of the corresponding representative point, relative to the value of the function itself, which determines a programme, etc.

On the other hand, the basic problem itself of determining the controlling moment under the action of which the motion of a body with given properties is possible has an ambiguous solution.

As a matter of fact, the projections of the required controlling moment contain the arbitrary function $\Phi(x_1, \ldots, x_6)$, as can be seen from (4.4) in particular, and from (4.3) in general. This helps us in solving the basic problem under the additional requirement that the programmed motion of the body under consideration be also its stable motion, in Lyapunov's sense, with respect to some given indices.

Suppose that the problem of determining the controlling moment is formulated in such a way that the programmed motion (4.1) of the body under consideration is asymptotically stable with respect to the property (4.1) itself.

The corresponding equation of perturbed motion in this case has the form

$$\dot{\omega}_3 = \Delta \cdot \Phi(x_1, \ldots, x_6) \quad (\Phi(x_1, \ldots, x_6)|_{\omega_3=0} = 0). \quad (5.1)$$

Its trivial solution $\omega_3 = 0$ corresponds to the programmed motion (4.1) of the body, which is taken as the unperturbed motion.

The arbitrary function $\Phi(x_1, \ldots, x_6)$, which eventually determines the controlling moment in accordance with (4.3), may be constructed, for example, in the form of a certain function of ω_3 and t, satisfying the inequality

$$\Delta \cdot \omega_3 \Phi(\omega_3, t) < 0 \quad (5.2)$$

in a certain ε-neighbourhood of the surface (4.1). In this case the asymptotic stability of the programmed motion

under consideration will be ensured if initial perturbations are present.

The problem of determining the controlling moment under the action of which the programmed motion of a heavy rigid body with one fixed point is one of its possible motions may also be formulated in conjunction with other modifications of the stability problems (stability of a programmed motion under continuous and parametric perturbations or stability over a finite interval of time), as well as with the problems of optimization of a programmed motion, the optimal motion of a representative point on the given surface, etc.

In conclusion, it may be mentioned that the above modifications of the inverse problems of dynamics of a rigid body may also be extended to cases when the body is situated in a Newtonian force field, in a potential field, or, in general, under the action of any arbitrary forces. In these cases, the construction of equations of programmed rotational motion of a rigid body around its centre of mass, together with the problems of stability and optimization, offers great promise for applications.

Appendix

CONSTRUCTION OF EQUATIONS OF PROGRAMMED MOTION IN GENERALIZED COORDINATES

Let us apply the methods described in Chs. 1 and 3 for constructing the equations of motion of mechanical systems in generalized coordinates in such a way that a given motion is one of the possible motions of the system under consideration.

1. Consider a mechanical system whose general equation of motion has the form

$$\sum_{\nu=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \frac{\partial T}{\partial q_\nu} - Q_\nu \right\} \delta q_\nu = 0, \qquad (1.1)$$

where $q\,[q_1, \ldots, q_n]$ is the vector of generalized coordinates; T and Q are the kinetic energy and generalized forces respectively, calculated by taking into consideration the holonomic constraints imposed beforehand on the system.

We require that one of the possible motions of the system under consideration be the motion with given properties

$$\Omega\colon\ \omega_\mu(q, t) = 0 \quad (\mu = 1, \ldots, m < n), \qquad (1.2)$$

where $\omega_\mu(q, t)$ are independent functions, which are bounded, continuous, and differentiable in a certain domain $G\,\{q\}$ for all $t \geqslant t_0$. On this basis, let us construct the equation of programmed motion of the system.

It should be noted that while constructing the required equations, we should treat the equations (1.2) as programmed constraints. Hence, the vector of the generalized velocity of motion of the corresponding representative point satisfies the following conditions on the manifold Ω (1.2) as well as outside it:

$$(\operatorname{grad}_q \omega_\mu \cdot \dot{q}) = \Phi_\mu(q, \omega, t) \quad (\mu = 1, \ldots, m), \qquad (1.3)$$

where Φ_μ (q, ω, t) are arbitrary functions which vanish when the programmed constraints (1.2) are satisfied.

The conditions (1.3) may be used for constructing the equations of programmed motion of control systems in the most general case, when the mathematical models of the system links, constraints imposed on the system, as well as the forces continuously acting on the body, are given beforehand. In the present case, a general equation of motion of the system has already been constructed by taking into consideration the holonomic unprogrammed constraints and continuously acting forces. It remains to construct the equations of programmed motion of this system by proceeding from the given general equation of motion (1.1) so that in the course of motion the generalized velocities of the system satisfy the conditions (1.3).

The solution of this problem is reduced to the embedding of the programmed constraints into the equations of motion of the system.

At the very outset, it should be remarked that the conditions (1.3), which must be satisfied by the generalized velocities in the programmed motion, are not integrable. Hence, in the present case, while computing the kinetic energy of the system, it is impossible to construct the equations of motion in generalized coordinates by directly imposing the conditions (1.3).

It should be noted that in analytical mechanics, the embedding of nonholonomic constraints on the equations of motion is justified while constructing these equations in Chaplygin's form, Boltzmann-Hamel form, Appel's form, and Schouten's form, when certain assumptions are made concerning the dynamic structure of the system, and the constraints themselves. In Rosenberg's book*, a method has been described for imposing linear nonholonomic constraints on Lagrange's equations in a quite general case. We shall be using this method while solving the problem formulated above, regarding the construction of equations of motion of programmed motion systems.

It follows from the formulation of the problem that the virtual displacements of the mechanical system under con-

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* Rosenberg, R. M. *Analytical Dynamics of Discrete Systems.* New York, London, 1977.

sideration satisfy the conditions

$$(\operatorname*{grad}_{q} \omega_\mu \cdot \delta q) = 0 \quad (\mu = 1, \ldots, m). \tag{1.4}$$

These conditions can be used to express the dependent variations δq_μ of coordinates in terms of independent ones δq_s ($\mu = 1, \ldots, m$, $s = m + 1, \ldots, n$):

$$\delta q_\mu = \sum_{s=m+1}^{n} c_{\mu s} \delta q_s \quad (\mu = 1, \ldots, m), \tag{1.5}$$

where $c_{\mu s} = c_{\mu s}(q, t)$ are the elements of the matrix

$$-\begin{Vmatrix} \frac{\partial \omega_1}{\partial q_1}, & \ldots, & \frac{\partial \omega_1}{\partial q_m} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial \omega_m}{\partial q_1}, & \ldots, & \frac{\partial \omega_m}{\partial q_m} \end{Vmatrix}^{-1} \begin{Vmatrix} \frac{\partial \omega_1}{\partial q_{m+1}}, & \ldots, & \frac{\partial \omega_1}{\partial q_n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial \omega_m}{\partial q_{m+1}}, & \ldots, & \frac{\partial \omega_m}{\partial q_n} \end{Vmatrix}.$$

The general equation of motion (1.1) of the system can be written in the form

$$\sum_{\mu=1}^{m} \mathcal{L}_\mu \delta q_\mu + \sum_{s=m+1}^{n} \mathcal{L}_s \delta q_s = 0, \tag{1.6}$$

where

$$\mathcal{L}_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right) - \frac{\partial T}{\partial q_\nu} - Q_\nu \quad (\nu = 1, \ldots, n).$$

Substituting (1.5) into (1.6), we get

$$\sum_{s=m+1}^{n} \Big(\sum_{\mu=1}^{m} c_{\mu s} \mathcal{L}_\mu + \mathcal{L}_s\Big) \delta q_s = 0. \tag{1.7}$$

The variations δq_s ($s = m + 1, \ldots, n$) are independent, hence

$$\sum_{\mu=1}^{m} c_{\mu s} \mathcal{L}_\mu + \mathcal{L}_s = 0 \quad (s = m + 1, \ldots, n). \tag{1.8}$$

Finally, we get the following equations for the programmed motion of the system under consideration:

$$\sum_{\mu=1}^{m} \left\{ c_{\mu s} \left[\frac{d}{dt} \left(\frac{\partial T}{d\dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu \right] \right\}$$

$$+ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} - Q_s = 0, \quad (1.9)$$

$$(s = m + 1, \ldots, n),$$

$$\sum_{\nu=1}^{n} \frac{\partial \omega_\mu}{\partial q_\nu} \dot{q}_\nu = \Phi_\mu (q, \ \omega, \ t) \quad (\mu = 1, \ldots, m).$$

These equations describe the motion of a system with the given properties Ω (1.2). Moreover, this motion is realized only when the initial state of the system satisfies the conditions

$$\omega_\mu (q_0, \ t_0) = 0 \ (\mu = 1, \ . \ . \ ., \ m). \quad (1.10)$$

It is natural to assume that the conditions (1.10) are not satisfied in actual practice. Hence, it is necessary to require in future that the motion of the system be stable (asymptotically stable) with respect to the given properties Ω (1.2) in the case of initial deviations from the conditions (1.10).

2. By way of an example, let us consider the motion of a material particle along a loxodrome.

The general equation of motion of a material point of mass m on the surface of the Earth has the following form in spherical coordinates:

$$\{mr \ [\ddot{\varphi} - (\dot{\psi} + \tau)^2 \sin \varphi \cdot \cos \varphi] - F_\varphi\} \ \delta\varphi$$

$$+ \{mr \ [\ddot{\psi} \sin \varphi + 2\dot{\varphi} \ (\dot{\psi} + \tau) \cos \varphi] - F_\psi\} \ \delta\psi = 0, \quad (2.1)$$

where φ and ψ are the spherical coordinates (ψ is the longitude); r and τ are the radius and the angular velocity of the Earth, and F_φ and F_ψ are the projections of the resultant force applied to the particle onto the tangents to the meridian and the parallel respectively.

We require that one of the possible motions of the particle be its motion along a loxodrome on the surface of the Earth:

$$\Omega: \quad \omega \equiv \ln \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} - (\psi - \psi_0)\cot\alpha = 0, \tag{2.2}$$

where φ_0 and ψ_0 are the given initial values of the spherical coordinates, α is the angle between the trajectory of the point and the Earth's meridians. Then, the corresponding programmed constraint is written in the form

$$\dot{\psi}\sin\varphi - \dot{\varphi}\tan\alpha = \Phi(\omega, \varphi, \psi, t), \tag{2.3}$$

where $\Phi(\omega, \varphi, \psi, t)$ is an arbitrary function satisfying the condition

$$\Phi(0, \varphi, \psi, t) = 0.$$

The virtual displacements of the point are given by

$$\sin\varphi\delta\psi - \tan\alpha\delta\varphi = 0. \tag{2.4}$$

The imposition of programmed constraints (2.3) on the general equation (2.1) leads to the following equations of programmed motion of the particle along a loxodrome on the Earth's surface:

$$\cot\alpha\sin\varphi\{mr[\ddot{\varphi} - (\dot{\psi} + \tau)^2\sin\varphi\cdot\cos\varphi] - F_\varphi\}$$
$$+ mr[\ddot{\psi}\sin\varphi + 2\dot{\varphi}(\dot{\psi} + \tau)\cdot\cos\varphi] - F_\psi = 0, \qquad (2.5)$$
$$\dot{\psi}\sin\varphi - \dot{\varphi}\tan\alpha = \Phi(\omega, \varphi, \psi, t). \qquad (2.5)$$

It should be observed that by an appropriate choice of the function Φ, we can ensure that the motion of the particle along a loxodrome on the Earth's surface will be stable with respect to the programme (2.2) if initial perturbations are present.

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